Asymptotics of quantum hyperbolic invariants

Stéphane Baseilhac, Institut Fourier

GdR Tresses "Braids in Paris", september 17-20, 2008

Plan of the talk

Plan of the talk

1. Volume conjecture(s) ?
2. Local model of q-Teichmüller spaces
3. Asymptotic integral formula
4. Main observation, and two questions

- $K \subset S^3$ knot
- $[S^3 \setminus K]_{hyp}$ hyperbolic pieces (JSJ decomposition)
- $J_r(K) \in \mathbb{Z}[q^{\pm 1}]$ normalized rth Jones polynomial

Volume conjecture (Kashaev 1997, Murakami-Murakami 1999)

$$2\pi \lim_{r\to\infty} \frac{\log |J_r(K)(e^{2\pi\sqrt{-1}/r})|}{r} = \mathrm{Vol}([S^3 \setminus K]_{hyp})$$

Quantum technology detects geometries ? ?

- $K \subset S^3$ knot
- $[S^3 \setminus K]_{hyp}$ hyperbolic pieces (JSJ decomposition)
- $J_r(K) \in \mathbb{Z}[q^{\pm 1}]$ normalized rth Jones polynomial

Volume conjecture (Kashaev 1997, Murakami-Murakami 1999)

$$2\pi \lim_{r\to\infty} \frac{\log |J_r(K)(e^{2\pi\sqrt{-1}/r})|}{r} = \mathrm{Vol}([S^3 \setminus K]_{hyp})$$

Quantum technology detects geometries ? ?

- $K \subset S^3$ knot
- $[S^3 \setminus K]_{hyp}$ hyperbolic pieces (JSJ decomposition)
- $J_r(K) \in \mathbb{Z}[q^{\pm 1}]$ normalized rth Jones polynomial

Volume conjecture (Kashaev 1997, Murakami-Murakami 1999)

$$2\pi \lim_{r \to \infty} \frac{\log |J_r(K)(e^{2\pi\sqrt{-1}/r})|}{r} = \mathrm{Vol}([S^3 \setminus K]_{hyp})$$

Quantum technology detects geometries ? ?

- $K \subset S^3$ knot
- $[S^3 \setminus K]_{hyp}$ hyperbolic pieces (JSJ decomposition)
- $J_r(K) \in \mathbb{Z}[q^{\pm 1}]$ normalized rth Jones polynomial

Volume conjecture (Kashaev 1997, Murakami-Murakami 1999)

$$2\pi \lim_{r\to\infty} \frac{\log |J_r(K)(e^{2\pi\sqrt{-1}/r})|}{r} = \mathrm{Vol}([S^3 \setminus K]_{hyp})$$

Quantum technology detects geometries ? ?

Quantum hyperbolic geometry

Theory of 3-dim spaces "locally modeled on $SL_q(2)$ reps"

For any

- aperiodic $\varphi \in \mathrm{Mod}(\Sigma_{g,n})$
- enhancement $\tilde{\rho}$ of $\rho \in \mathrm{Hom}^{inj}(\pi_1(\Sigma_{g,n}), PSL_2\mathbb{C})$

$\exists$ Quantum hyperbolic invts :

$$\mathcal{H}_r(\tilde{\rho})(\varphi) \in GL(\mathbb{C}^{r(4g-4+2n)})$$

Asymptotical behaviour as $r \to \infty$?

Plan of the talk

Fix $\zeta_r \in \mathbb{C}$, $(\zeta_r)^r = 1$

Weyl algebra ("quantum torus") :

$$\mathcal{W}_{\zeta_r} = \frac{\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]}{(XY = \zeta_r YX)}$$

Classical/quantum correspondance

$$\{ \mathcal{W}_{\zeta_r}\text{-irreps } \alpha \} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} (x^{1/r}, y^{1/r}\} \in \mathbb{C}^* \times \mathbb{C}^* \\ \alpha(X^r) = x \cdot Id, \alpha(Y^r) = y \cdot Id \end{array} \right\}$$

Fix $\zeta_r \in \mathbb{C}$, $(\zeta_r)^r = 1$

Weyl algebra ("quantum torus") :

$$\mathcal{W}_{\zeta_r} = \frac{\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]}{(XY = \zeta_r YX)}$$

Classical/quantum correspondance

$$\{\ \mathcal{W}_{\zeta_r}\text{-irreps } \alpha\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} (x^{1/r}, y^{1/r}\} \in \mathbb{C}^* \times \mathbb{C}^* \\ \alpha(X^r) = x \cdot Id, \alpha(Y^r) = y \cdot Id \end{array} \right\}$$

$$\mathbb{l} = \left(\frac{\log|x|}{r}, \frac{\log|y|}{r}\right) : \{P(x,y) = 0\} \longrightarrow \mathbb{R}^2$$

$$Image(\mathbb{l}) \longrightarrow \cup\{rays\ in\ \mathbb{R}^2\}, \quad r \to \infty$$

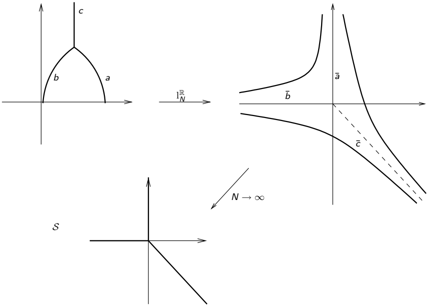

FIG.: $y = (1-x)^{-1}$

Plan of the talk

Recall

$$\mathcal{H}_r(\tilde{\rho}) : Mod(\Sigma_{g,n}) \to GL(\mathbb{C}^{r(4g-4+2n)})$$

Take 1-cusped M_φ :

$$M_\varphi = \frac{\Sigma_{g,n} \times [0,1]}{(x,0) \sim (\varphi(x),1)}$$

The invariants

$$\mathcal{H}_r(M_\varphi) := \mathrm{Trace}\mathcal{H}_r(\cdot)(\varphi)$$

define rational maps

$$\mathcal{H}_r(M_\varphi) : \chi_r \dashrightarrow \mathbb{C}$$

where

- $\pi_r : \chi_r \longrightarrow \chi$ degree r^2 covering
- χ d.f. comp^t of $\mathrm{Hom}(\pi_1(M_\varphi), PSL_2\mathbb{C})//PSL_2\mathbb{C}$

$\chi \hookrightarrow \mathbb{C}^* \times \mathbb{C}^*$ plane curve
χ_r defined via rth roots

The invariants

$$\mathcal{H}_r(M_\varphi) := \ \mathrm{Trace}\mathcal{H}_r(\cdot)(\varphi)$$

define rational maps

$$\mathcal{H}_r(M_\varphi) : \chi_r \dashrightarrow \mathbb{C}$$

where

- $\pi_r : \chi_r \longrightarrow \chi$ degree r^2 covering
- χ d.f. comp^t of $\mathrm{Hom}(\pi_1(M_\varphi), PSL_2\mathbb{C})//PSL_2\mathbb{C}$

$\chi \hookrightarrow \mathbb{C}^* \times \mathbb{C}^*$ plane curve
χ_r defined via rth roots

The invariants

$$\mathcal{H}_r(M_\varphi) := \mathrm{Trace}\mathcal{H}_r(\cdot)(\varphi)$$

define rational maps

$$\mathcal{H}_r(M_\varphi) : \chi_r \dashrightarrow \mathbb{C}$$

where

- $\pi_r : \chi_r \longrightarrow \chi$ degree r^2 covering
- χ d.f. comp^t of $\mathrm{Hom}(\pi_1(M_\varphi), PSL_2\mathbb{C})//PSL_2\mathbb{C}$

$\chi \hookrightarrow \mathbb{C}^* \times \mathbb{C}^*$ plane curve
χ_r defined via rth roots

The invariants

$$\mathcal{H}_r(M_\varphi) := \mathrm{Trace}\mathcal{H}_r(\cdot)(\varphi)$$

define rational maps

$$\mathcal{H}_r(M_\varphi) : \chi_r \dashrightarrow \mathbb{C}$$

where

- $\pi_r : \chi_r \longrightarrow \chi$ degree r^2 covering
- χ d.f. comp^t of $\mathrm{Hom}(\pi_1(M_\varphi), PSL_2\mathbb{C})//PSL_2\mathbb{C}$

$\chi \hookrightarrow \mathbb{C}^* \times \mathbb{C}^*$ plane curve
χ_r defined via rth roots

$\mathcal{H}_r(M_\varphi) : \chi_r \dashrightarrow \mathbb{C}$ satisfy :

Theorem

- $\chi_\infty := \lim \mathrm{Haussdorff}\{\chi_r\} \cong \cup\{rays\ in\ \mathbb{R}^2\} \times T^2$
- $\mathcal{H}_r(M_\varphi)(x_r)$ determined by $\lim x_r = x$ as $r \to \infty$:

$$\mathcal{H}_r(M_\varphi)(x_r) \propto_{r\to\infty} \int_{T^n} e^{\frac{r}{2i\pi}\mathcal{V}(z;x)} \frac{\wedge^i e^{-z_i} dz_i}{(e^{rz_i}-1)}$$

$\mathcal{H}_r(M_\varphi) : \chi_r \dashrightarrow \mathbb{C}$ satisfy :

Theorem

- $\chi_\infty := \lim \mathrm{Haussdorff}\{\chi_r\} \cong \cup\{rays\ in\ \mathbb{R}^2\} \times T^2$
- $\mathcal{H}_r(M_\varphi)(x_r)$ determined by $\lim x_r = x$ as $r \to \infty$:

$$\mathcal{H}_r(M_\varphi)(x_r) \propto_{r\to\infty} \int_{T^n} e^{\frac{r}{2i\pi}\mathcal{V}(z;x)} \frac{\wedge^i e^{-z_i} dz_i}{(e^{rz_i} - 1)}$$

Similar results for *any* manifold :

Corollary

For any link $L \subset S^3$, the quantum hyperbolic invariants $\mathcal{H}_r(L)$ grow at most polynomially with r

Plan of the talk

Main observation

$\{\lim_{r\to\infty} \frac{\log|\mathcal{H}_r(M_\varphi)(x_r)|}{r}\}$ is finite over "constant" sequences (x_r)

Conjecture

One of these values is $Vol(M_\varphi)$

Main observation

$\{\lim_{r\to\infty} \frac{\log |\mathcal{H}_r(M_\varphi)(x_r)|}{r}\}$ is finite over "constant" sequences (x_r)

Conjecture

One of these values is $Vol(M_\varphi)$

Questions

- Interpretation of integral $\int_{T^n} e^{\frac{r}{2i\pi}\mathcal{V}(z;x)} \frac{\wedge^i e^{-z_i} dz_i}{(e^{rz_i}-1)}$ in full ?
- Role for asymptotics of $J_r(K)$ near $e^{2i\pi/r}$, as $r \to \infty$?