Chord Slides and Universal Invariants

Joint w/ J.E. Andersen, J-B. Meilhan, and R.C. Penner

Tresses á Paris

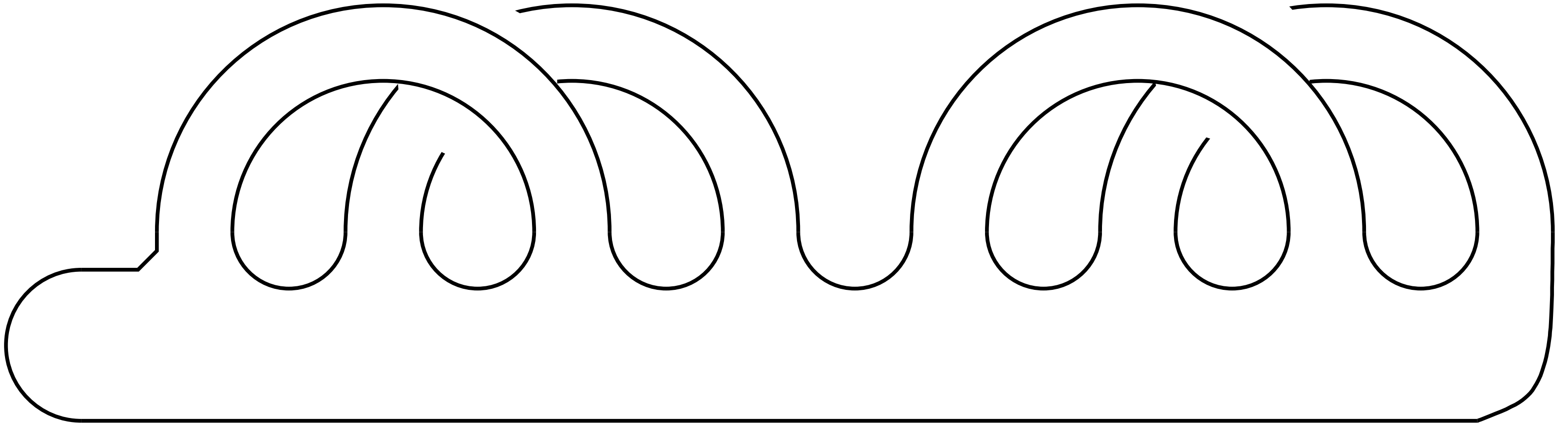
Alex James Bene

bene@usc.edu

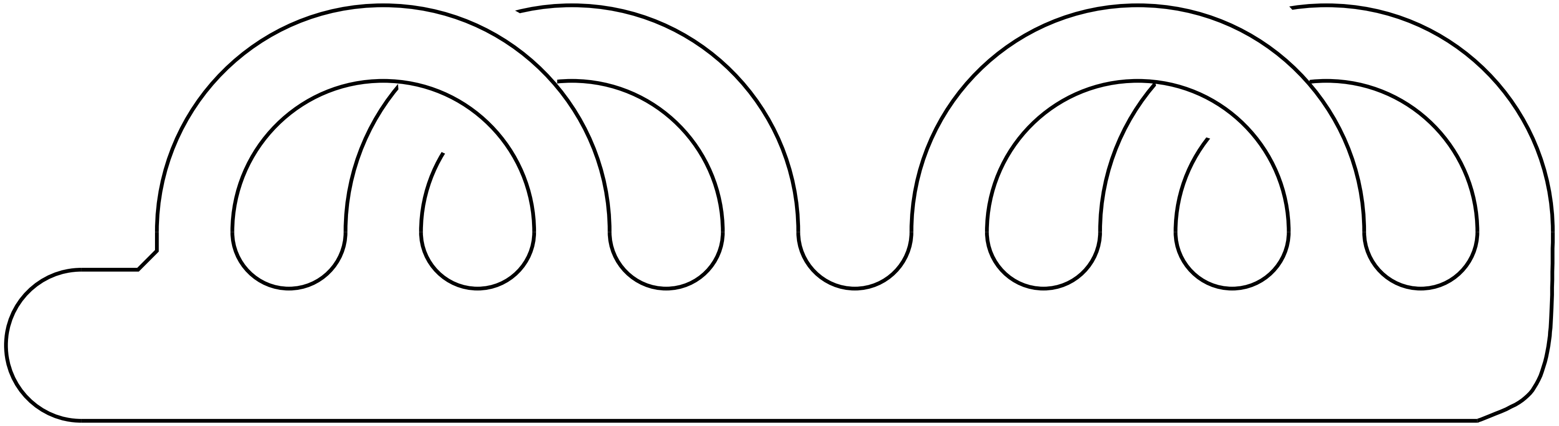
University of Southern California

CTQM

Some Notation



Some Notation



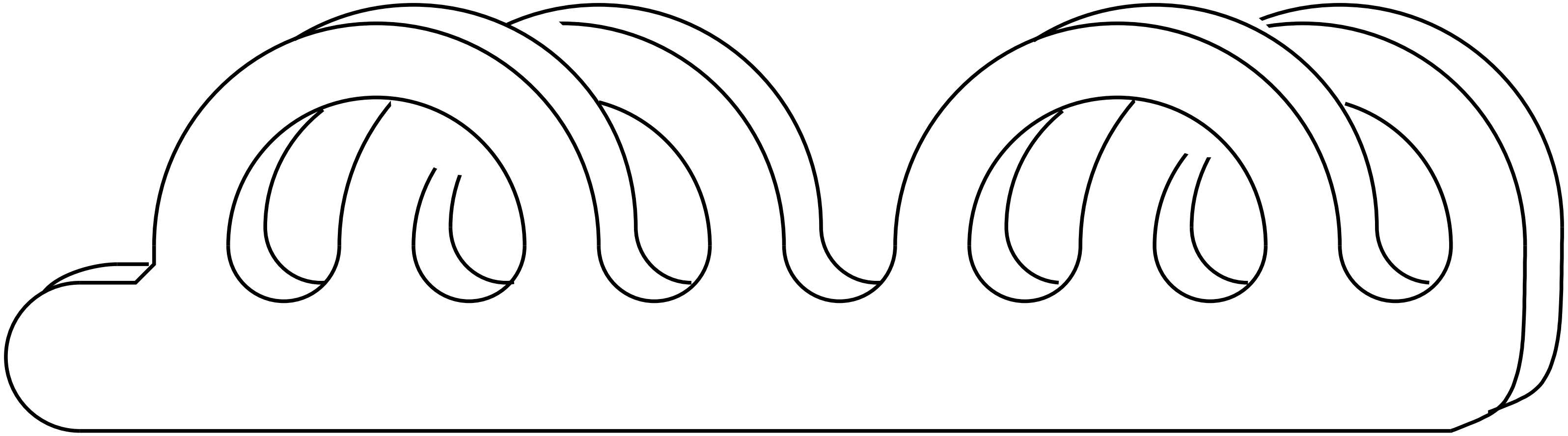
$\Sigma_{g,1} =$ genus g surface w/ one boundary component.

$$MC_{g,1} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,1}, \partial\Sigma_{g,1}))$$

$=$ mapping class group of $\Sigma_{g,1}$.

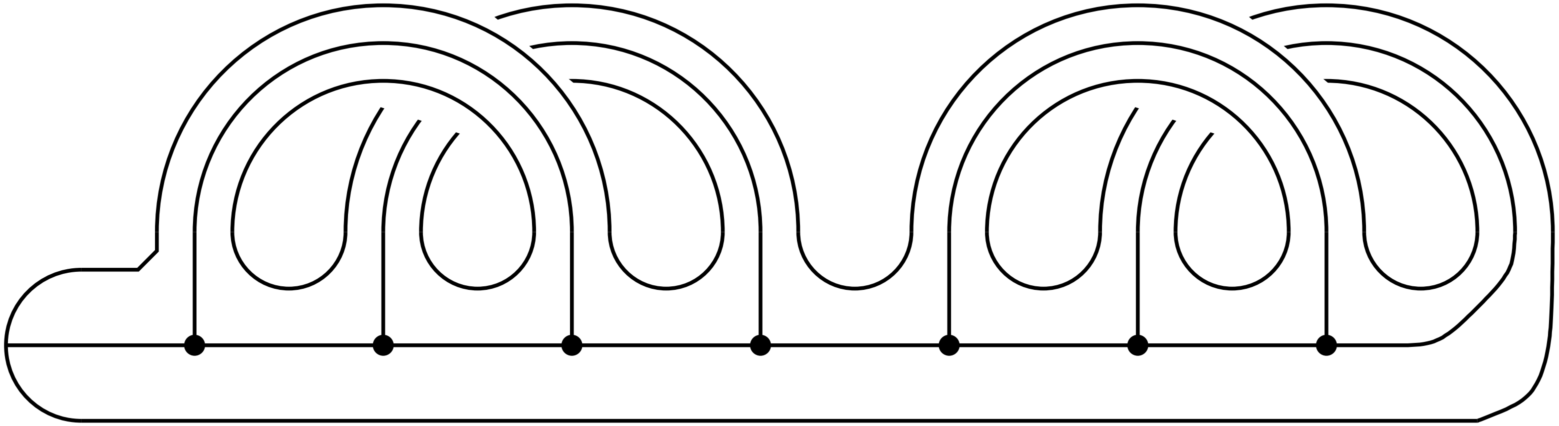
$$\pi = \pi_1(\Sigma_{g,1}, p), \quad H = \frac{\pi}{[\pi, \pi]}, \quad H_{\mathbb{Q}} = H \otimes \mathbb{Q}.$$

Some Notation



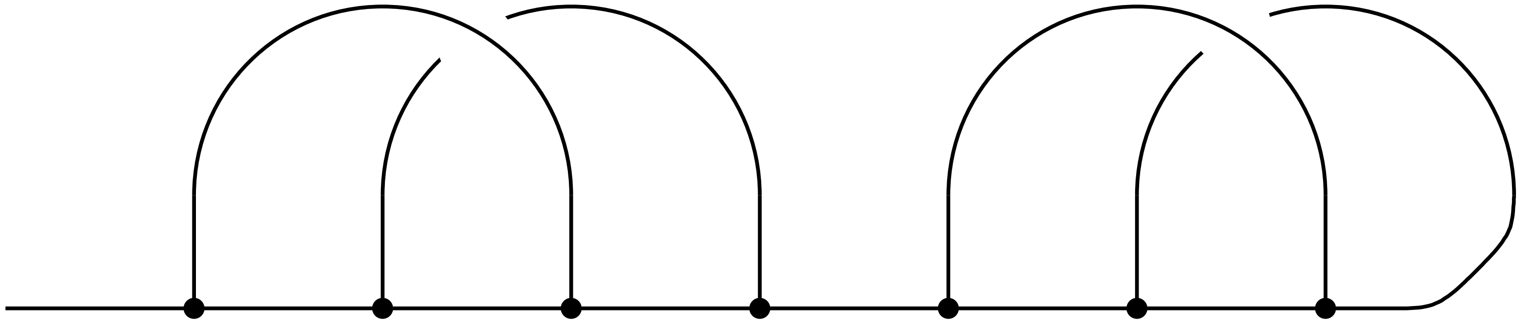
$1_{\Sigma_{g,1}}$ = the thickened surface $\Sigma_{g,1} \times I$.

Some Notation



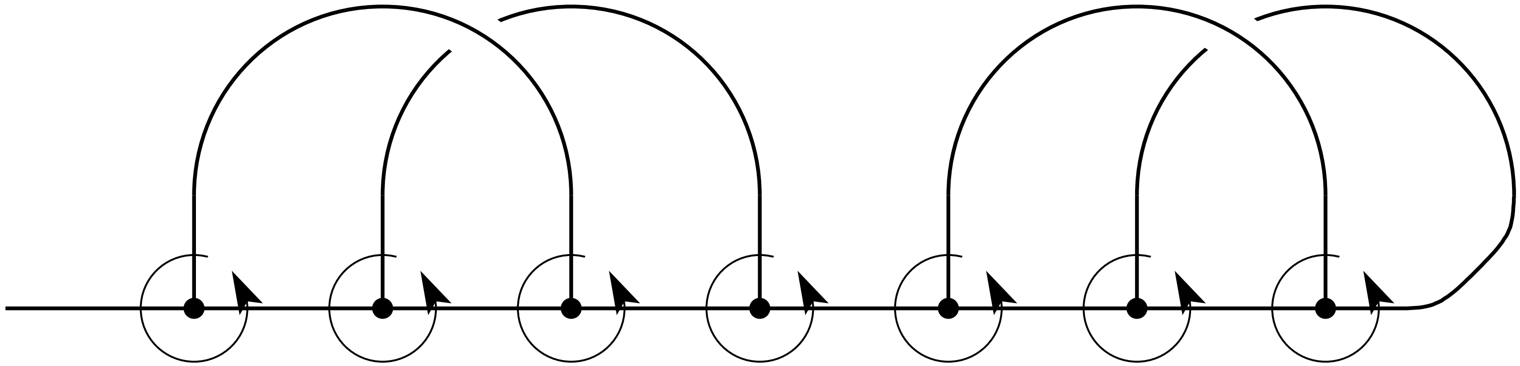
Combinatorially,...

Some Notation



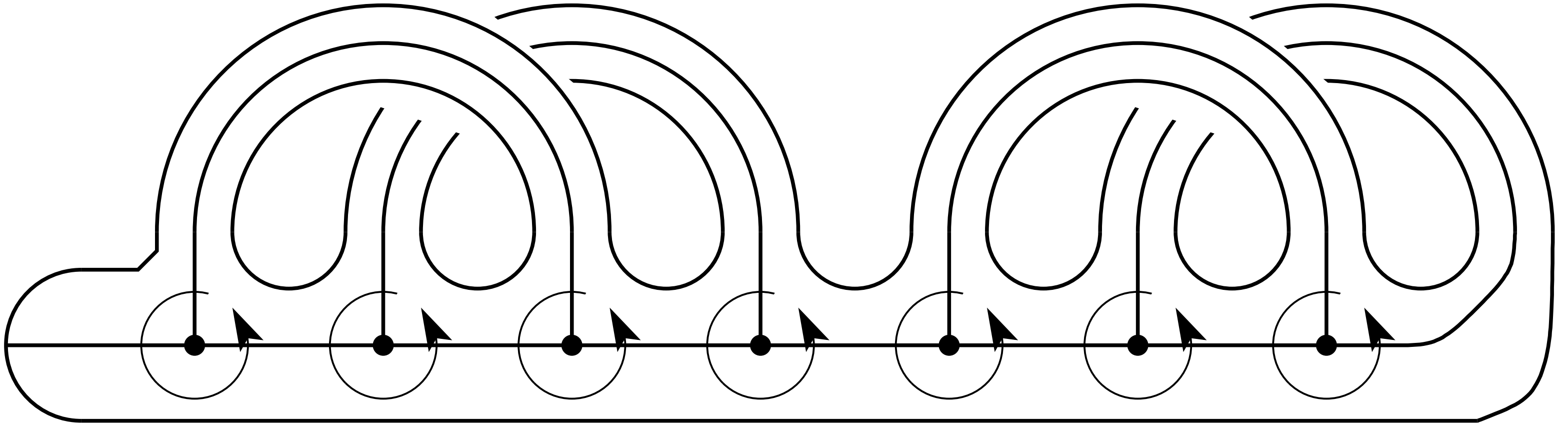
Combinatorially, we have a linear chord diagram G, consisting of a straight line *core* and $2g$ *chords*.

Some Notation



Or, a (linear) *bordered fatgraph*: a trivalent vertex-oriented graph with a uni-valent tail.

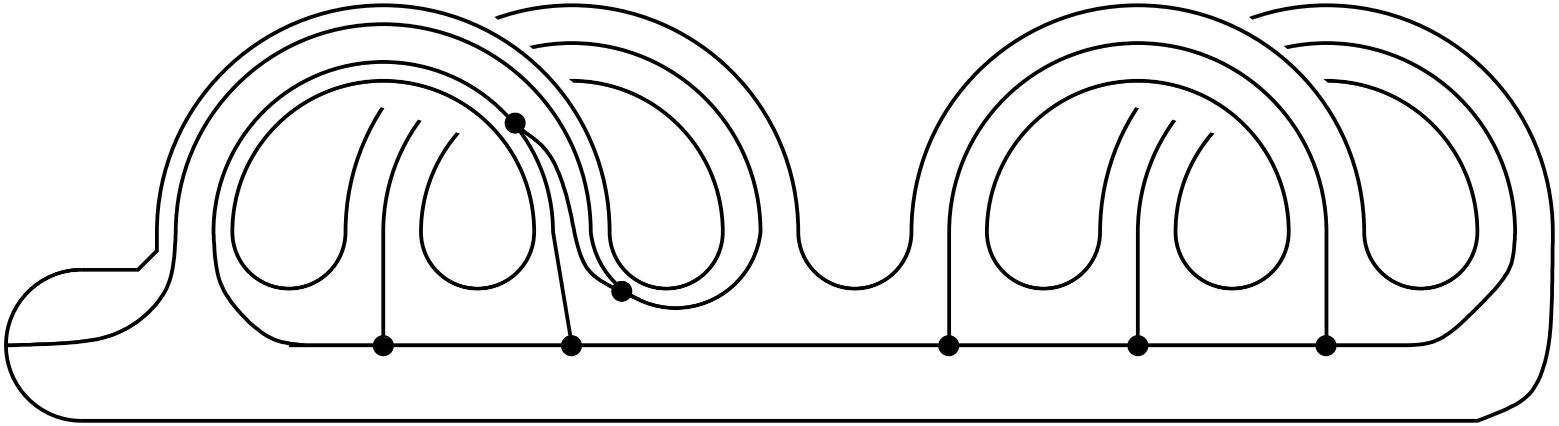
Some Notation



Or, a (linear) *bordered fatgraph*: a trivalent vertex-oriented graph with a uni-valent tail.

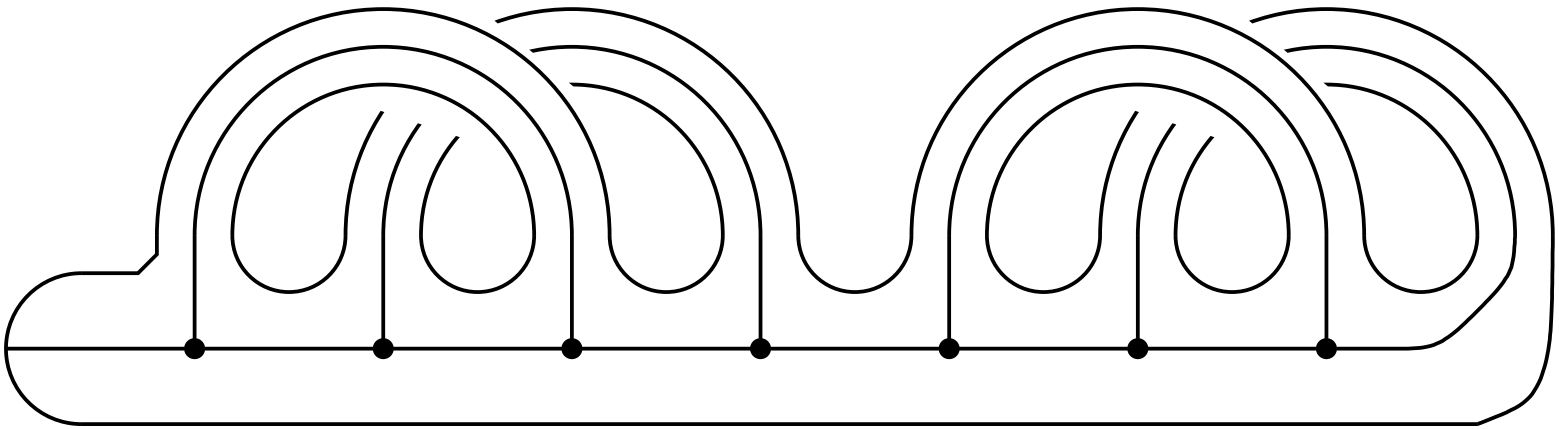
An embedding of a bordered fatgraph G into $\Sigma_{g,1}$ as a spine is called a *marking* of G. Let G^{st} denote the "standard" marked fatgraph shown above.

Some Notation



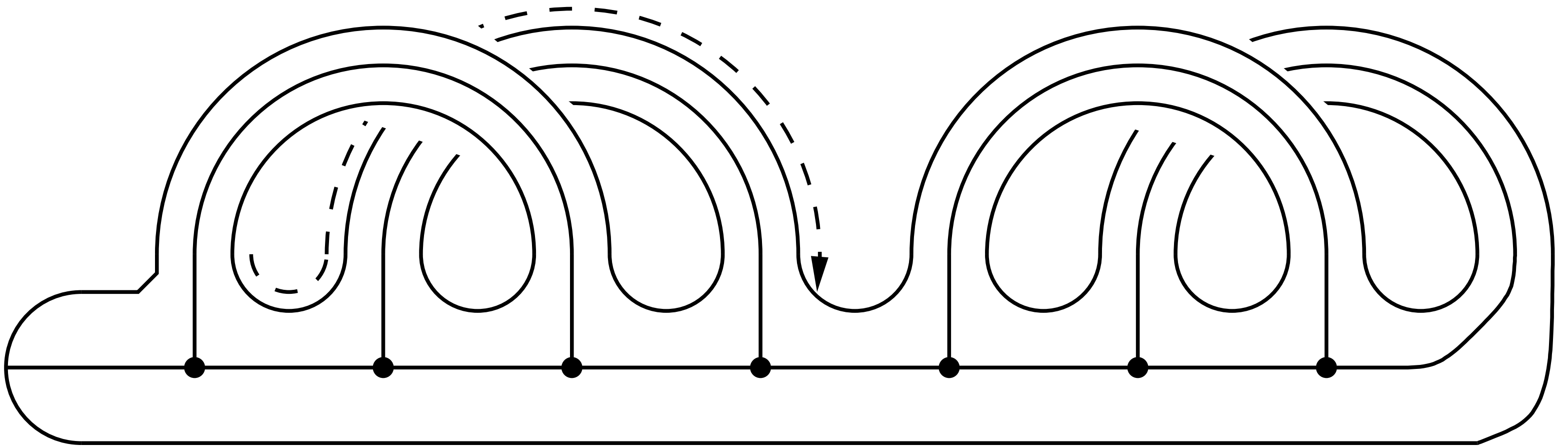
Or, a (linear) *bordered fatgraph*: a trivalent vertex-oriented graph with a uni-valent tail.
An embedding of a bordered fatgraph G into $\Sigma_{g,1}$ as a spine is called a *marking* of G.
$MC_{g,1}$ acts on the set of markings of G.

Chord Slides



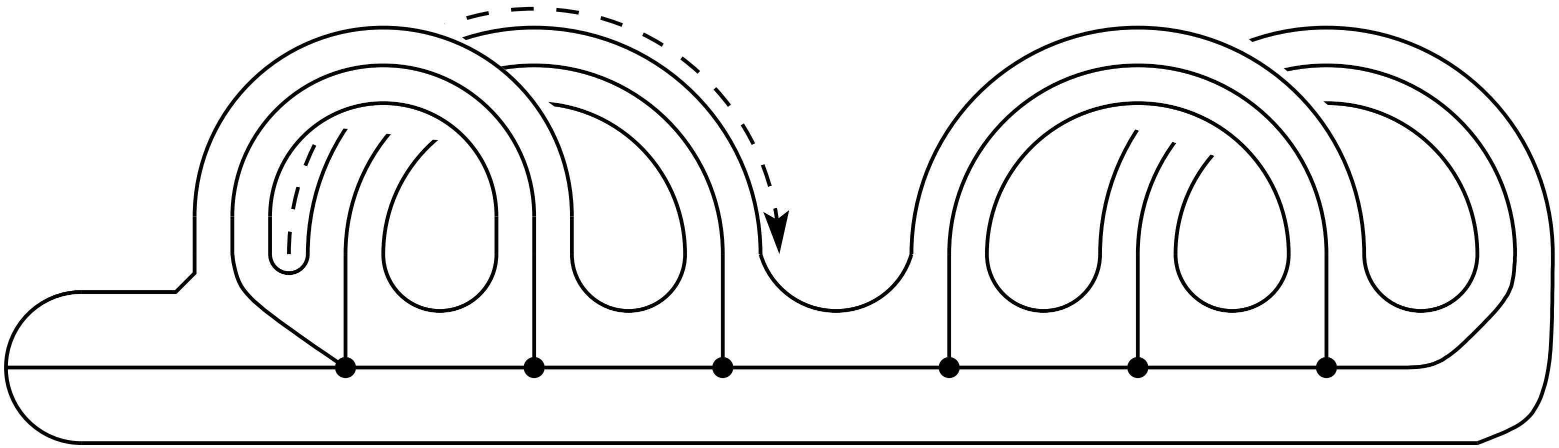
How can this help us understand the mapping class group?

Chord Slides



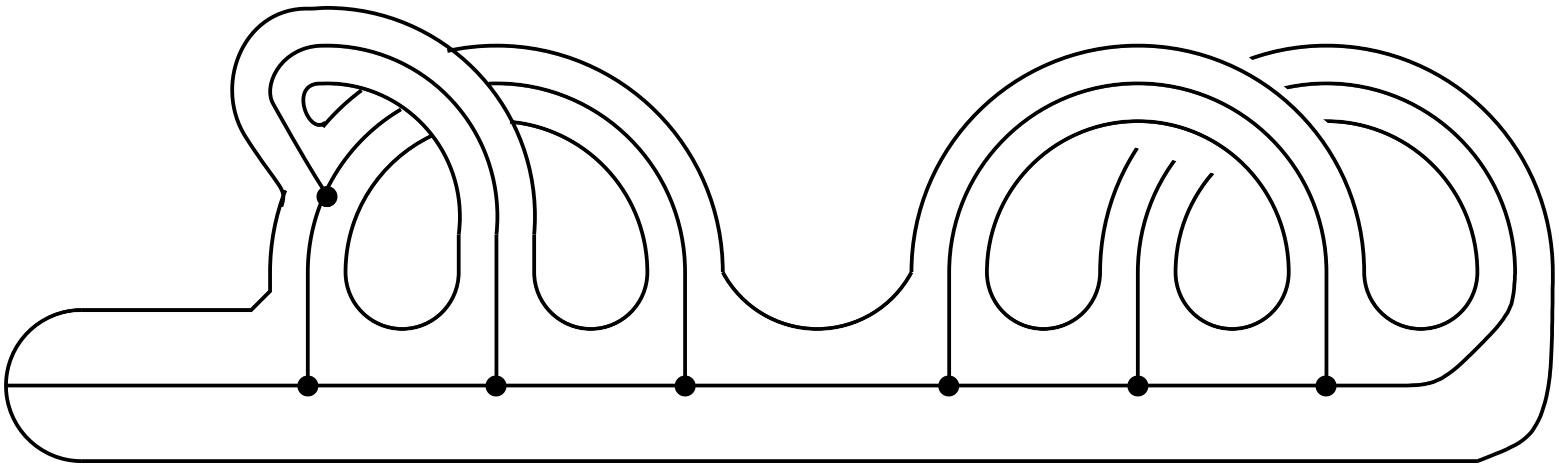
Consider the elementary move: the *chord slide*...

Chord Slides



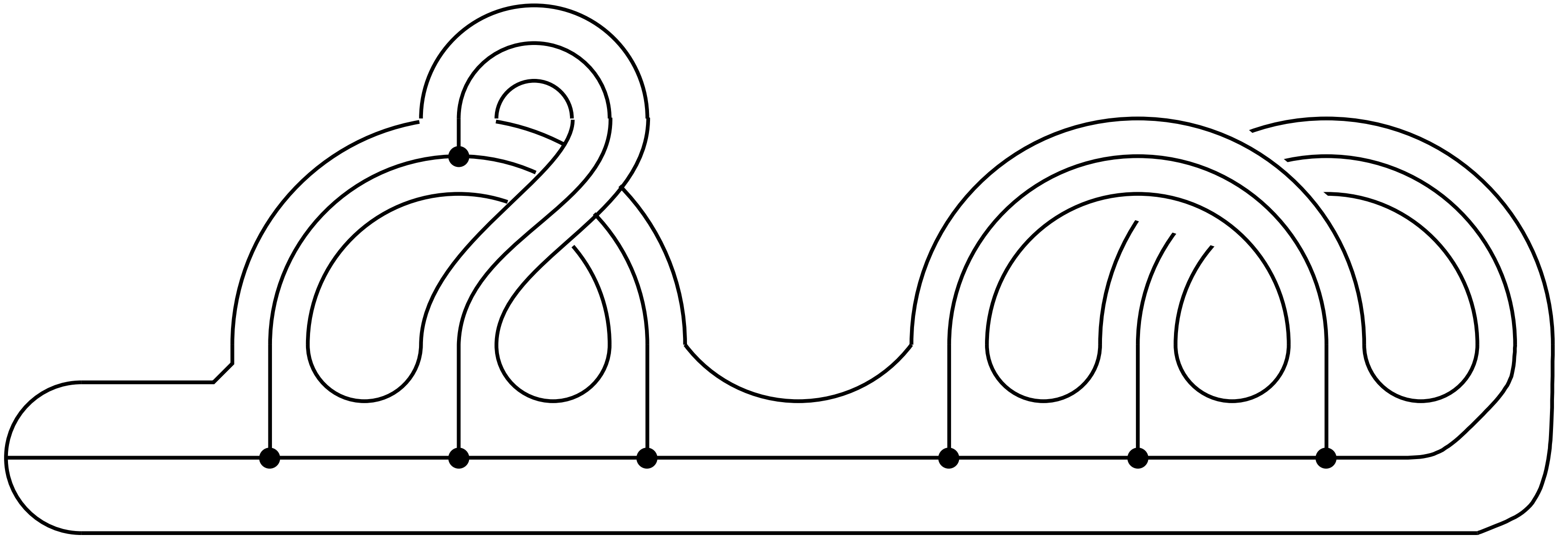
Consider the elementary move: the *chord slide*...

Chord Slides



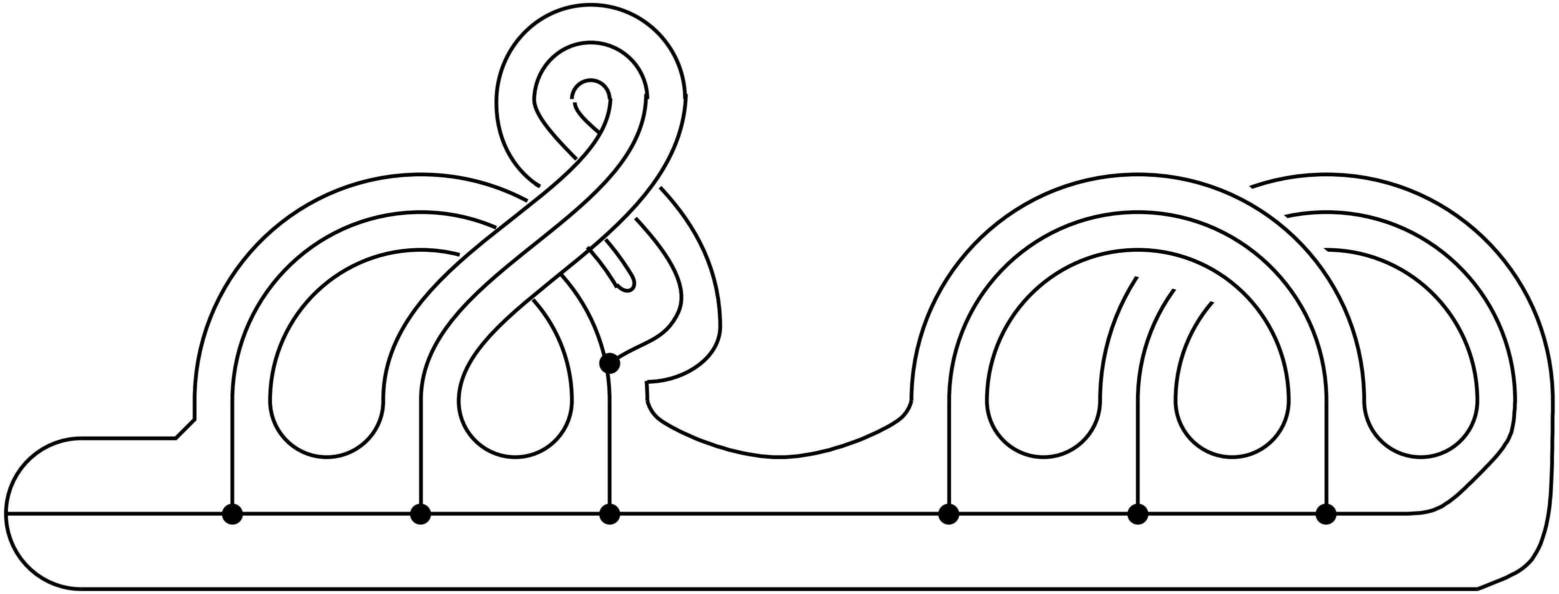
Consider the elementary move: the *chord slide*...

Chord Slides



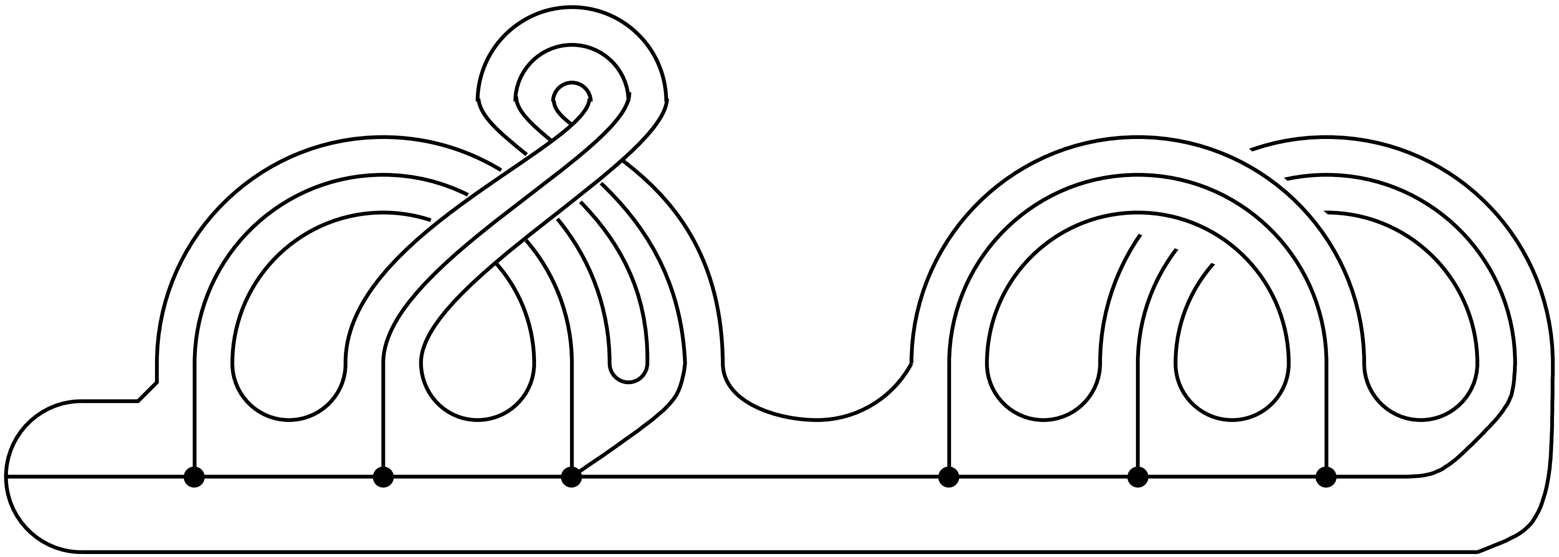
Consider the elementary move: the *chord slide*...

Chord Slides



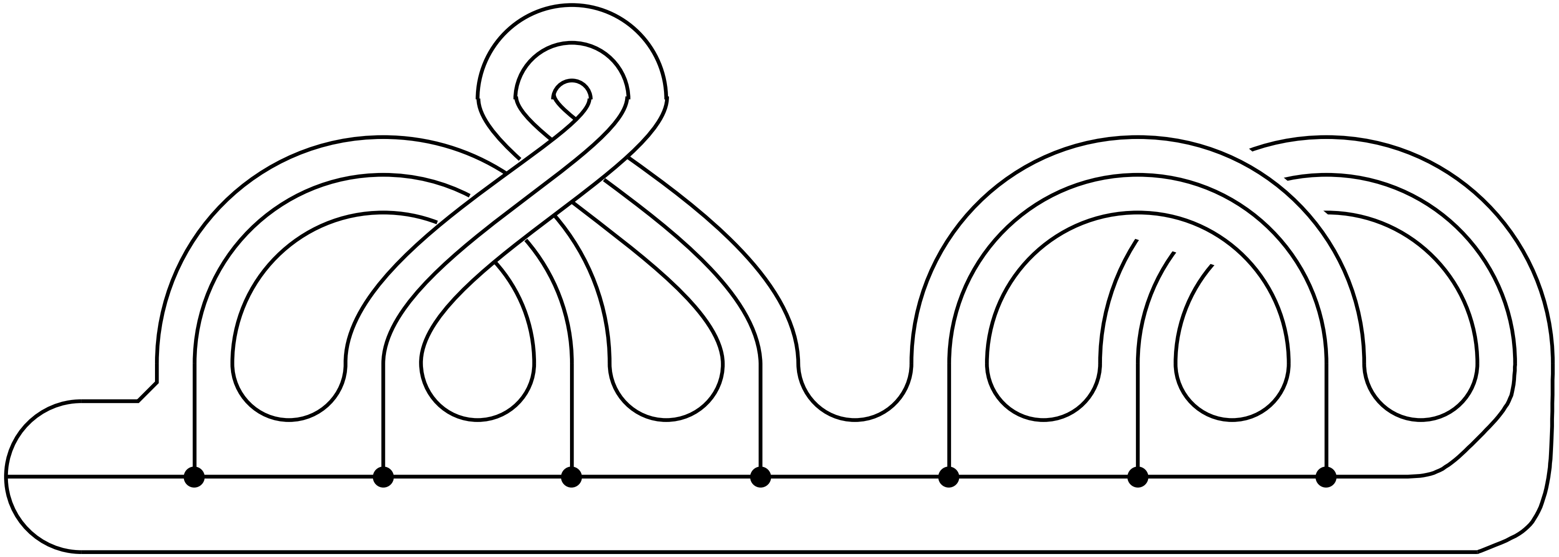
Consider the elementary move: the *chord slide*...

Chord Slides



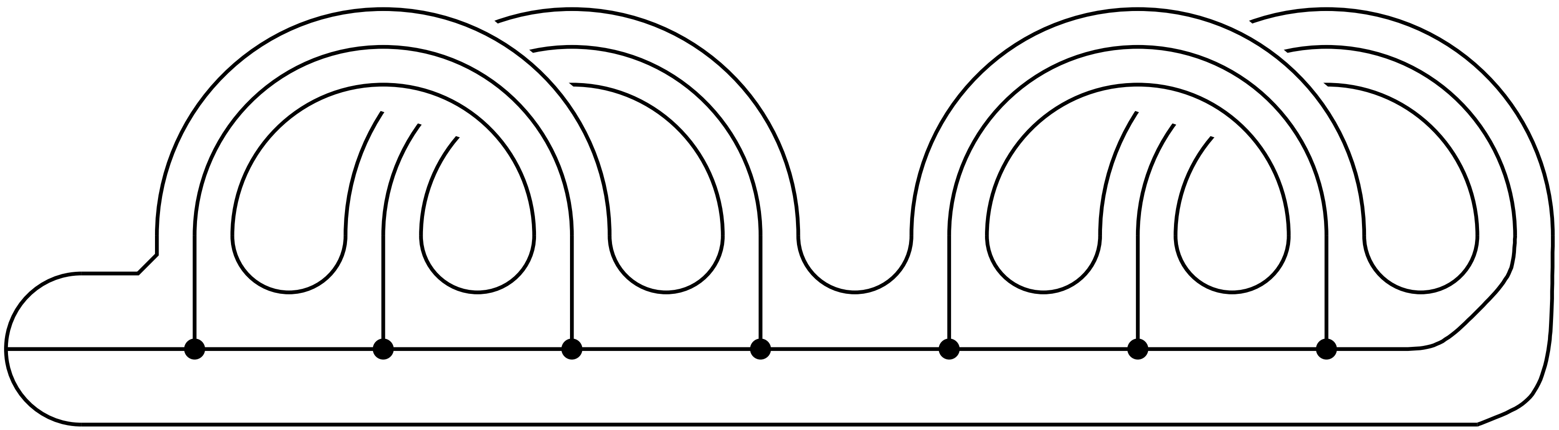
Consider the elementary move: the *chord slide*...

Chord Slides



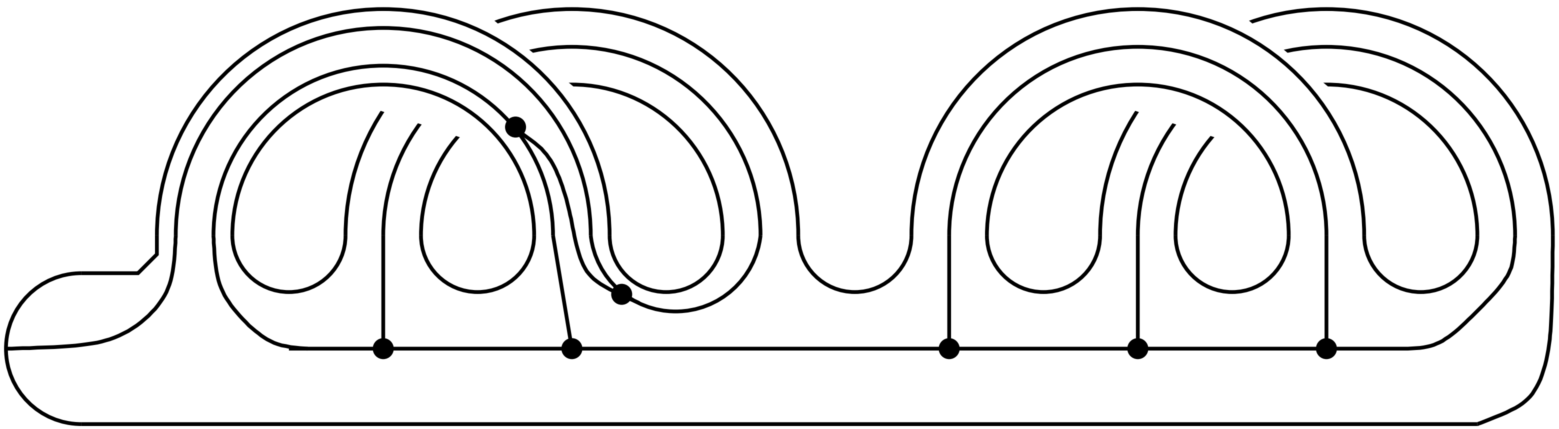
Consider the elementary move: the *chord slide*...

Chord Slides



Consider the elementary move: the *chord slide*...
Once we return to the same combinatorial graph, this defines a diffeomorphism of $\Sigma_{g,1}$ to itself, thus an element of the mapping class group $MC_{g,1}$.

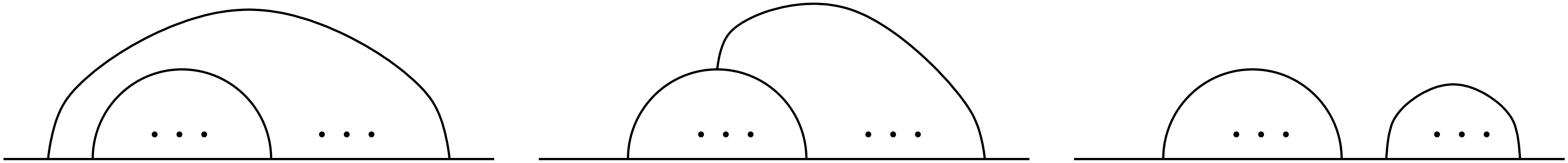
Chord Slides



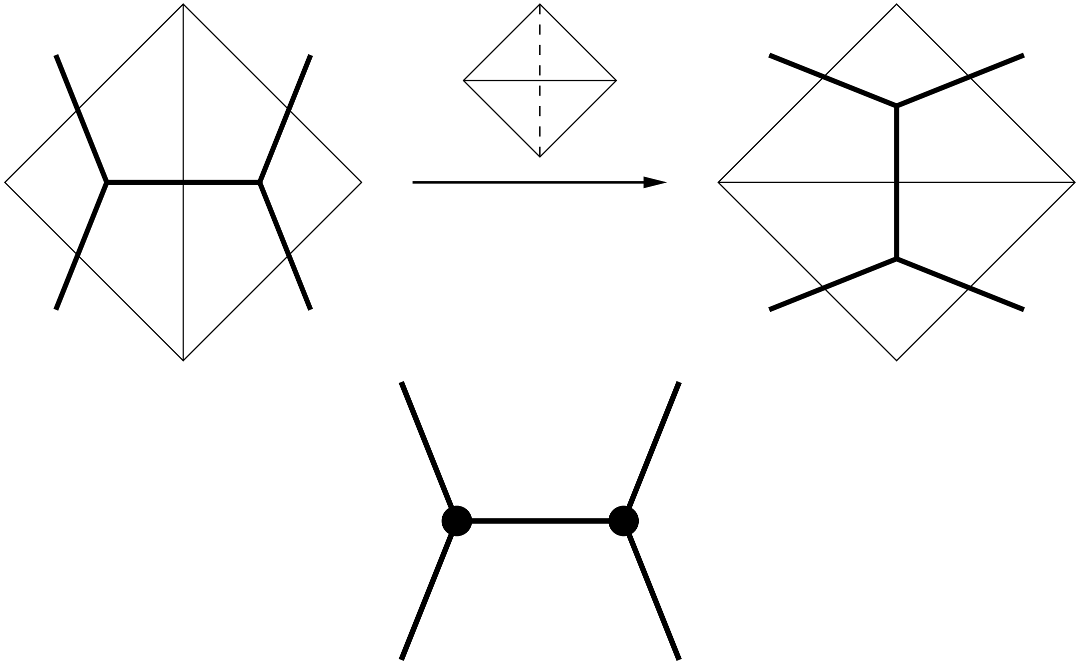
Consider the elementary move: the *chord slide*...
The image of the original fatgraph is now twisted.

Whitehead moves

A chord slide

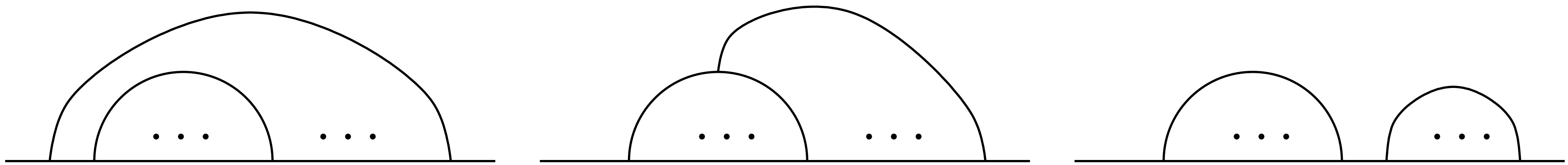


is comprised of two elementary *Whitehead moves* on the underlying fatgraph:

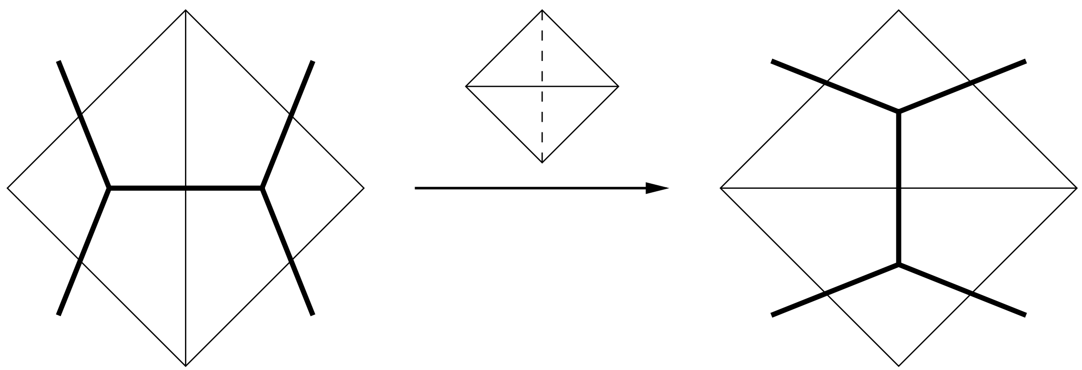


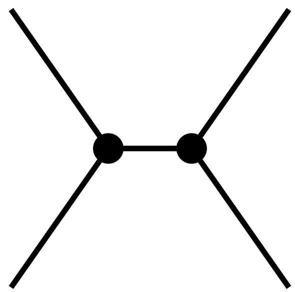
Whitehead moves

A chord slide



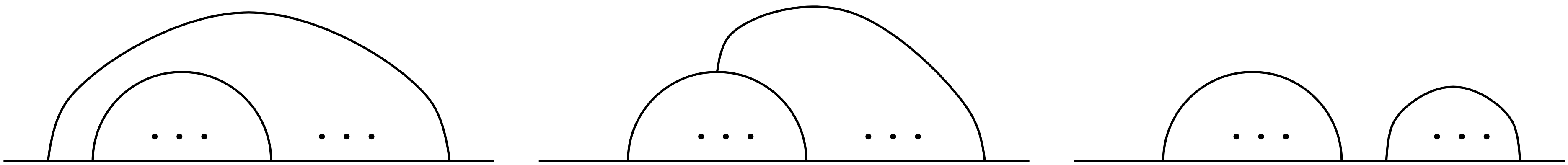
is comprised of two elementary *Whitehead moves* on the underlying fatgraph:



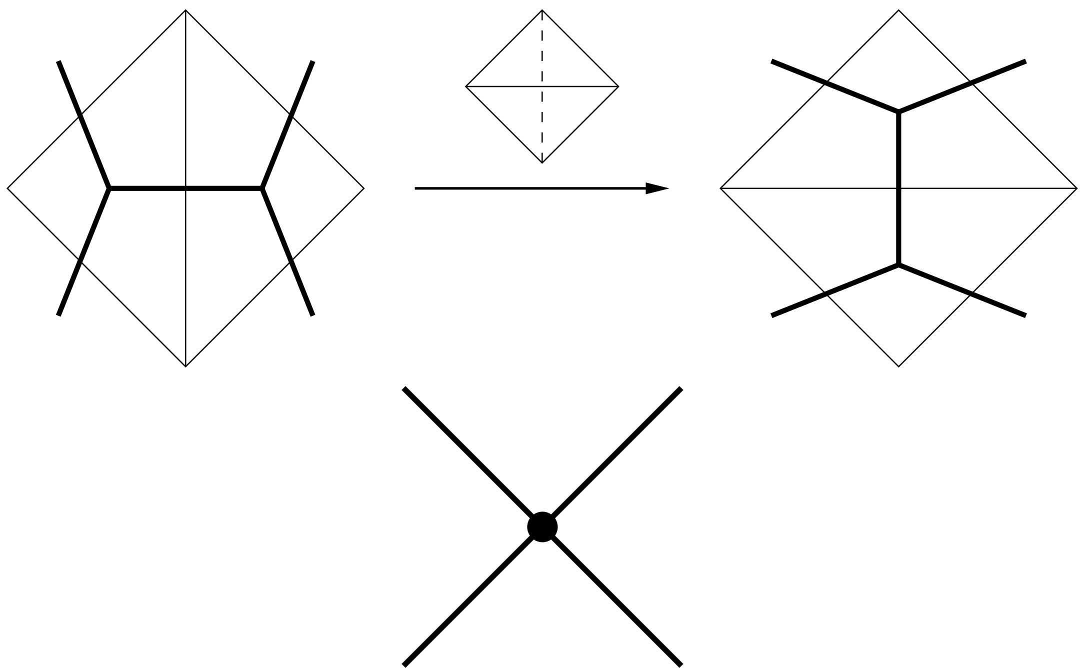


Whitehead moves

A chord slide

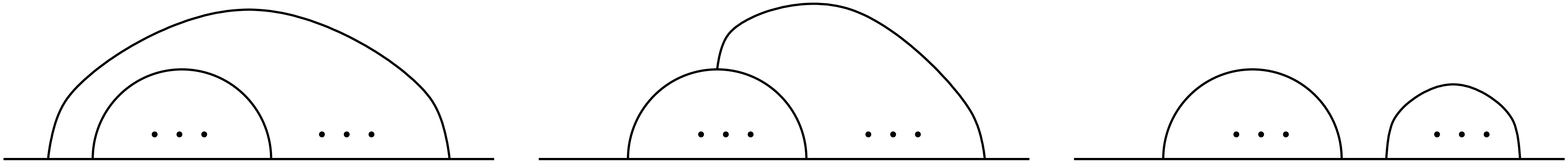


is comprised of two elementary *Whitehead moves* on the underlying fatgraph:

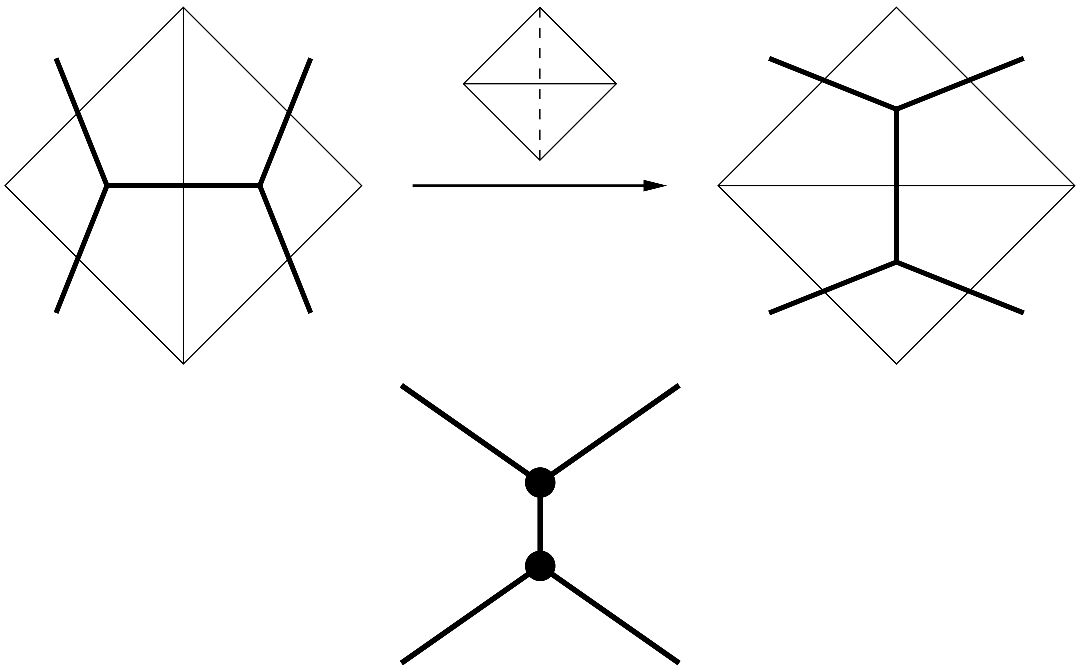


Whitehead moves

A chord slide

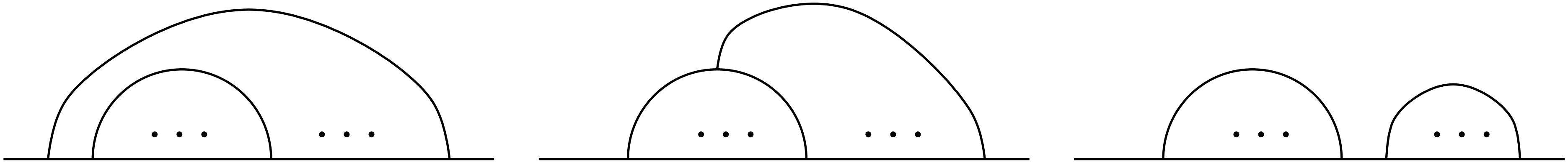


is comprised of two elementary *Whitehead moves* on the underlying fatgraph:

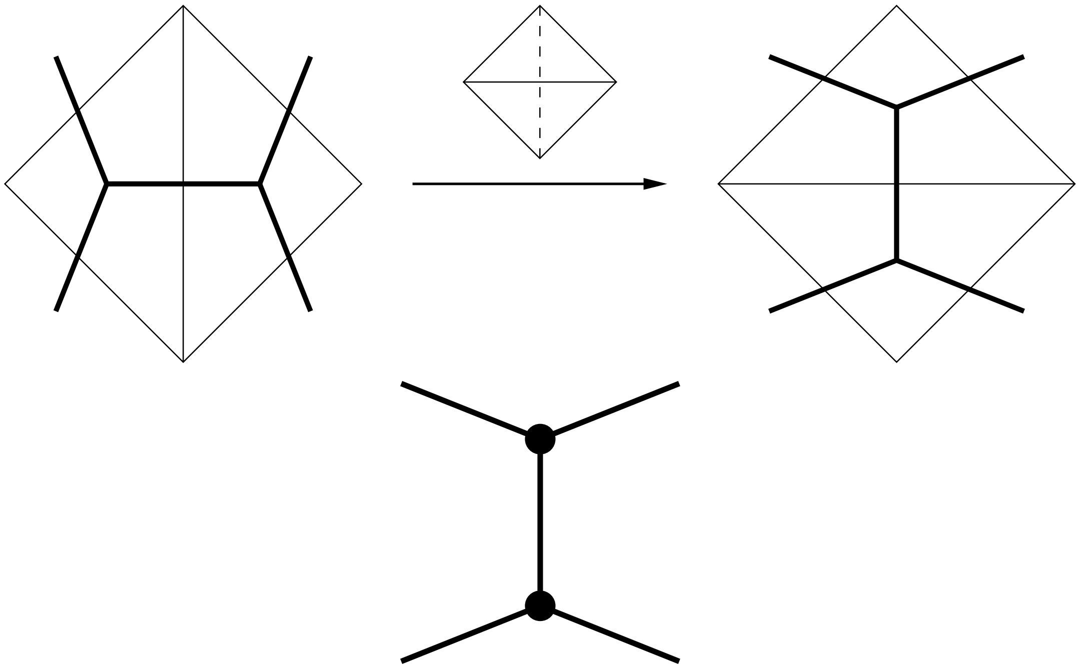


Whitehead moves

A chord slide



is comprised of two elementary *Whitehead moves* on the underlying fatgraph:



Theorems and Groupoids

Theorem (B)**.** *Any element of* $MC_{g,1}$ *is represented by a sequence of chord slides. Moreover, relations are precisely known.*

$\Rightarrow$ The chordslide groupoid of $\Sigma_{g,1}$

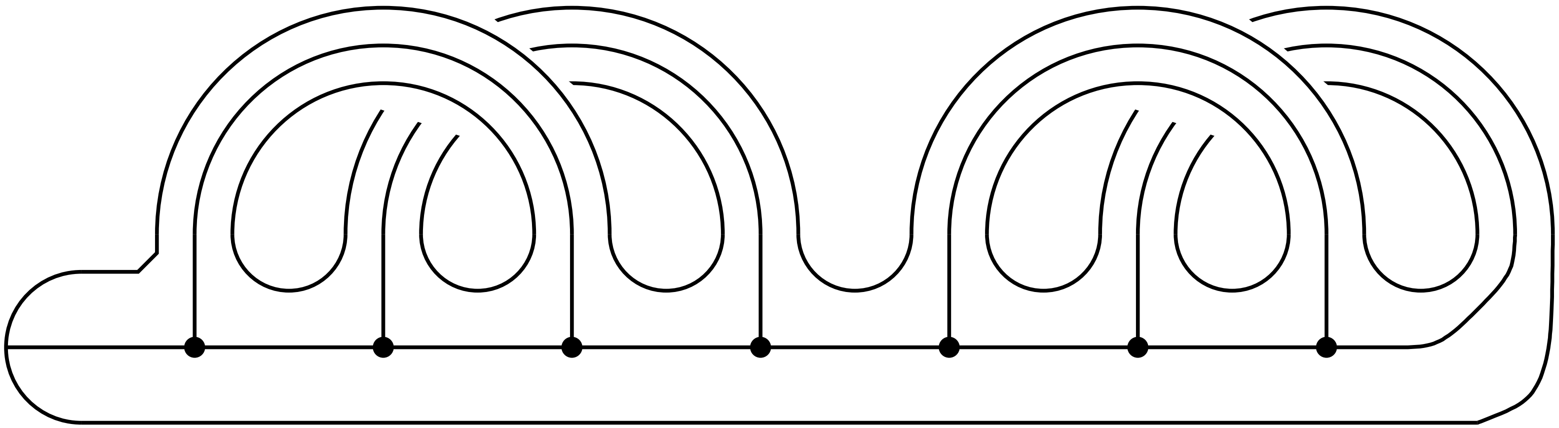
The proof uses the "chord diagram – fatgraph correspondence" and essentially relies on the fundamental results of:

Theorem (Whitehead, Penner, Harer-Strebel)**.** *Any element of* $MC_{g,1}$ *is represented by a sequence of Whitehead moves. Moreover, relations are precisely known.*

$\Rightarrow$ The Ptolemy groupoid of $\Sigma_{g,1}$

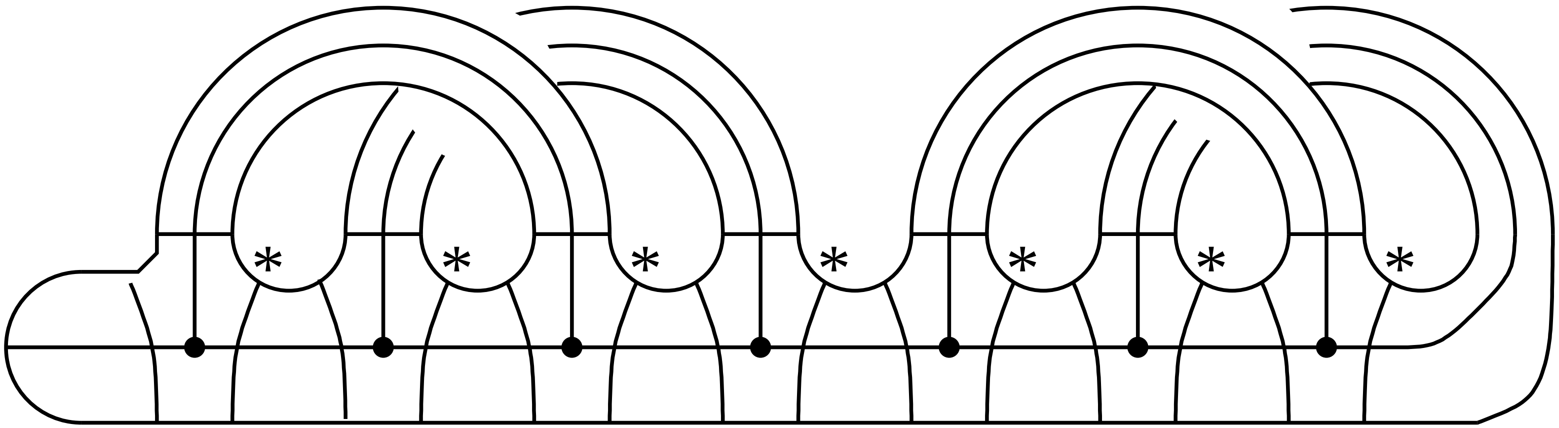
Tile decomposition

The combinatorial depiction of the surface also gives a decomposition of $\Sigma_{g,1}$ into tiles:



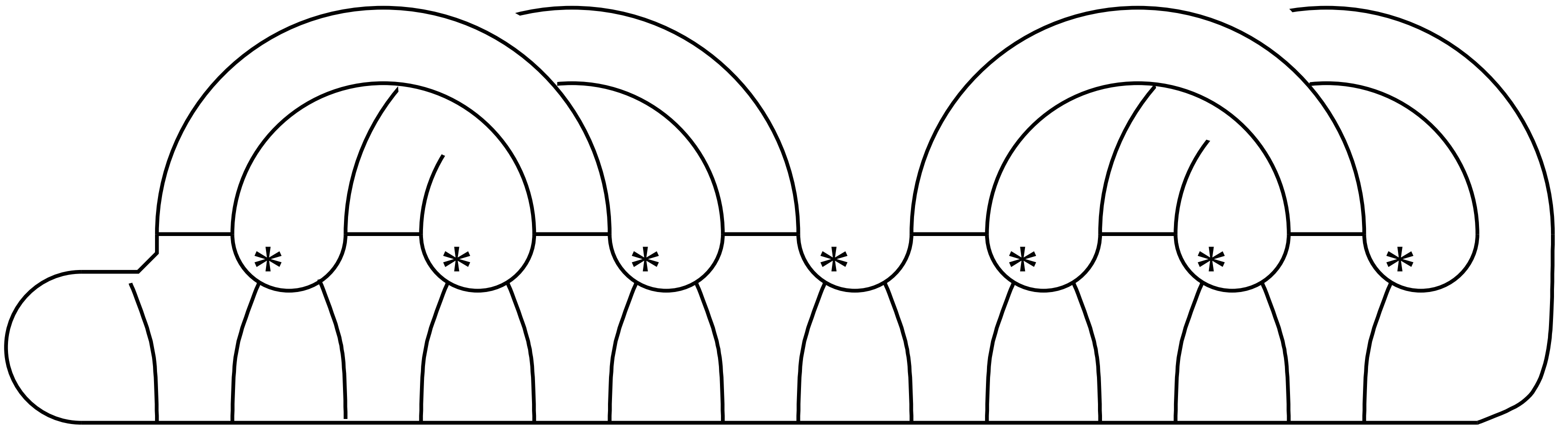
Tile decomposition

The combinatorial depiction of the surface also gives a decomposition of $\Sigma_{g,1}$ into tiles:



Tile decomposition

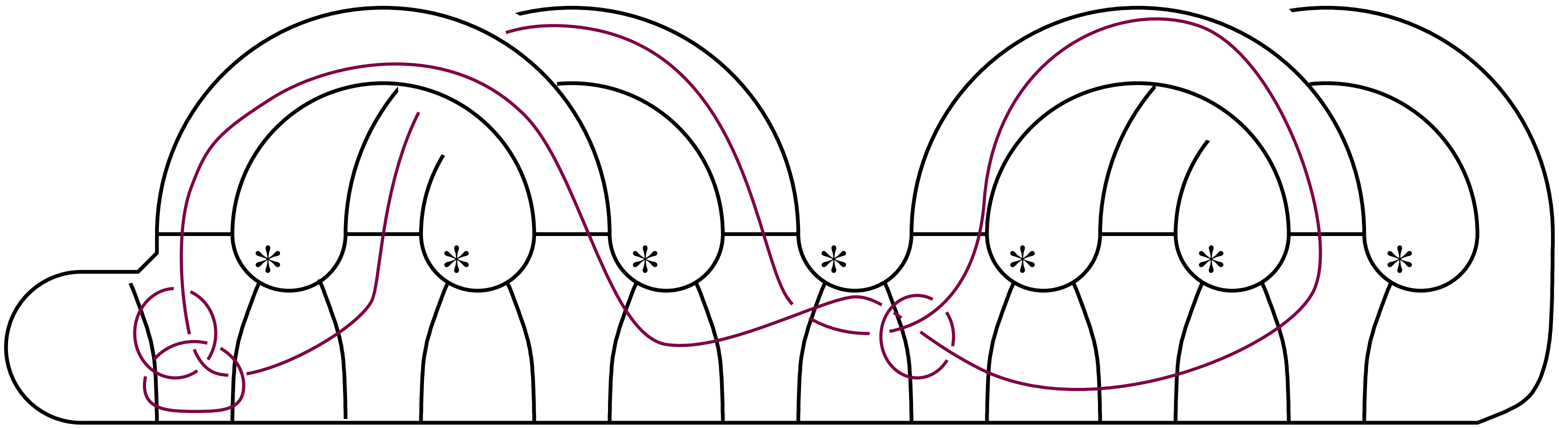
The combinatorial depiction of the surface also gives a decomposition of $\Sigma_{g,1}$ into tiles:



Consisting of Hexagons with a “forbidden sector”, squares, and a bi-gon...

Tile decomposition

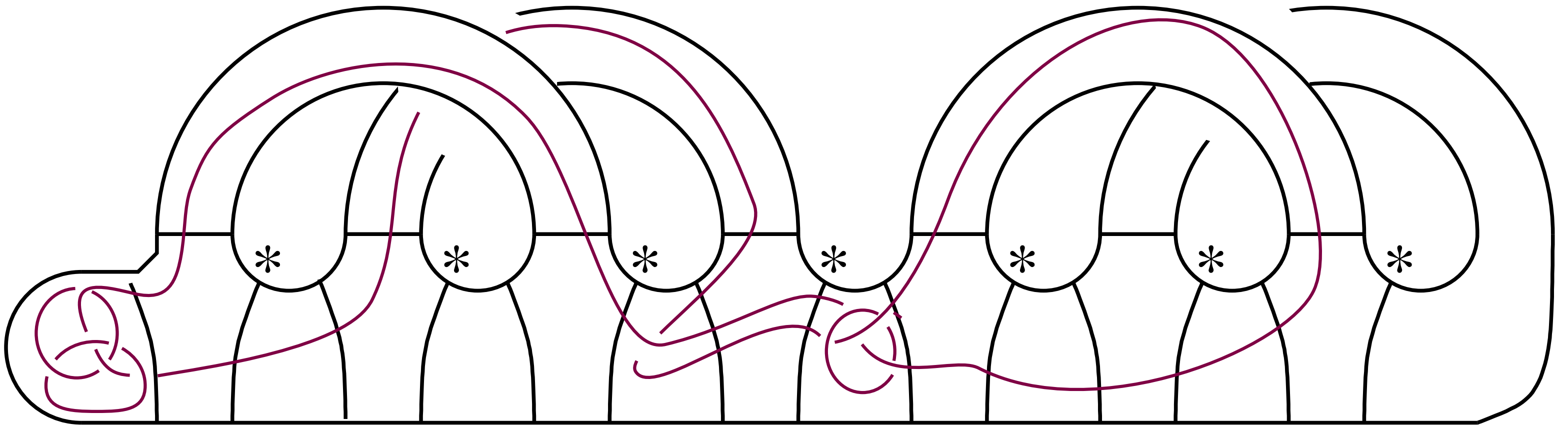
The combinatorial depiction of the surface also gives a decomposition of $\Sigma_{g,1}$ into tiles:



Consisting of Hexagons with a “forbidden sector”, squares, and a bi-gon, such that any link L in $1_{\Sigma_{g,1}}$ ...

Tile decomposition

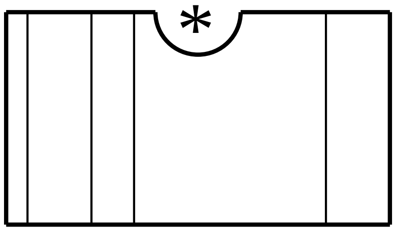
The combinatorial depiction of the surface also gives a decomposition of $\Sigma_{g,1}$ into tiles:



Consisting of Hexagons with a "forbidden sector", squares, and a bi-gon, such that any link L in $1_{\Sigma_{g,1}}$ can be isotoped so that:

AMR Structure

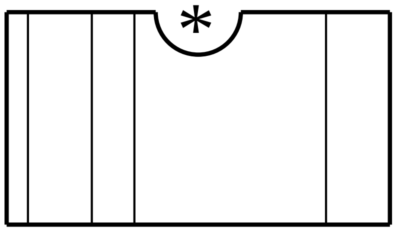
Any link L in $1_{\Sigma_{g,1}}$ can be isotoped so that:



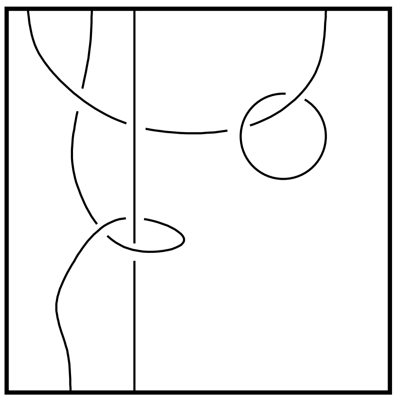
Each hexagon contains a trivial tangle avoiding the forbidden sector.

AMR Structure

Any link L in $1_{\Sigma_{g,1}}$ can be isotoped so that:



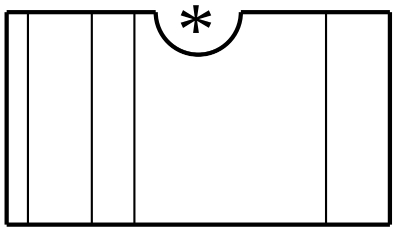
Each hexagon contains a trivial tangle avoiding the forbidden sector.



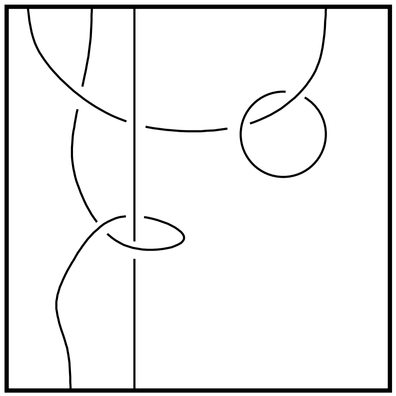
Squares can contain an arbitrary tangle,

AMR Structure

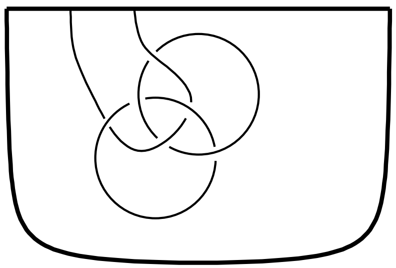
Any link L in $1_{\Sigma_{g,1}}$ can be isotoped so that:



Each hexagon contains a trivial tangle avoiding the forbidden sector.



Squares can contain an arbitrary tangle,



as can the bi-gon.

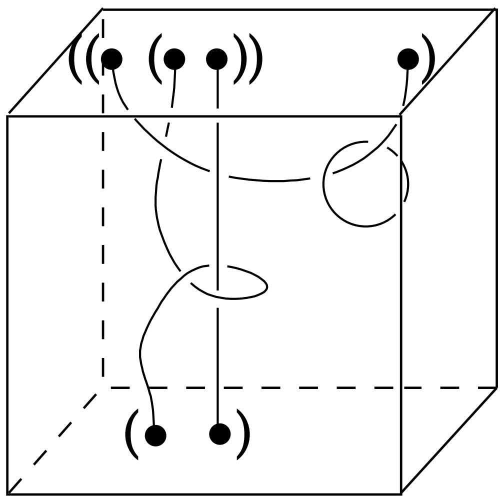
AMR Structures

This will allow us to define a (simplification of the) isotopy invariant of framed oriented links in $1_{\Sigma_{g,1}}$ introduced by Andersen, Mattes, and Reshetikhin: the AMR invariant.

But first, we must review the Kontsevich integral, which is an isotopy invariant of oriented non-associative tangles.

Non-associative tangles

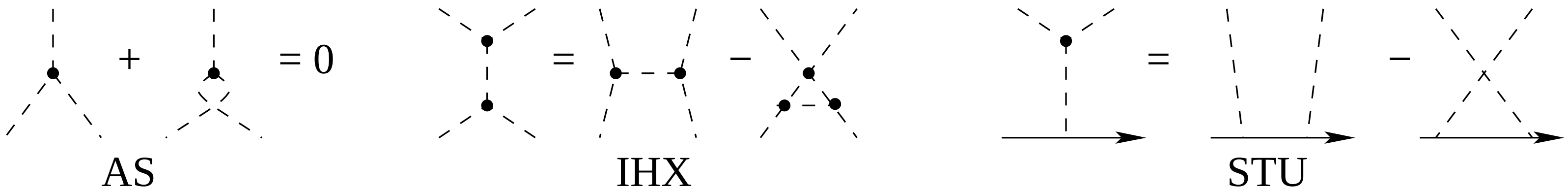
A *non-associative tangle* T is an embedding of a 1-manifold X into the standard cube C such that all boundary points of X are linearly arranged and *bracketed* on the top and the bottom of C.

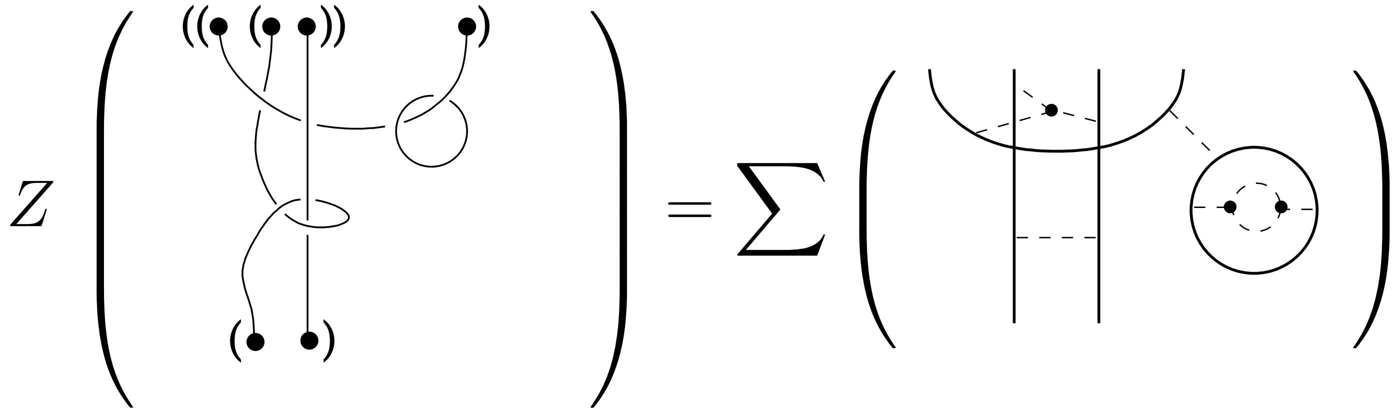


Kontsevich Integral

For an oriented non-associated tangle $T\colon X \hookrightarrow C$, the Kontsevich integral Z of T takes values in:

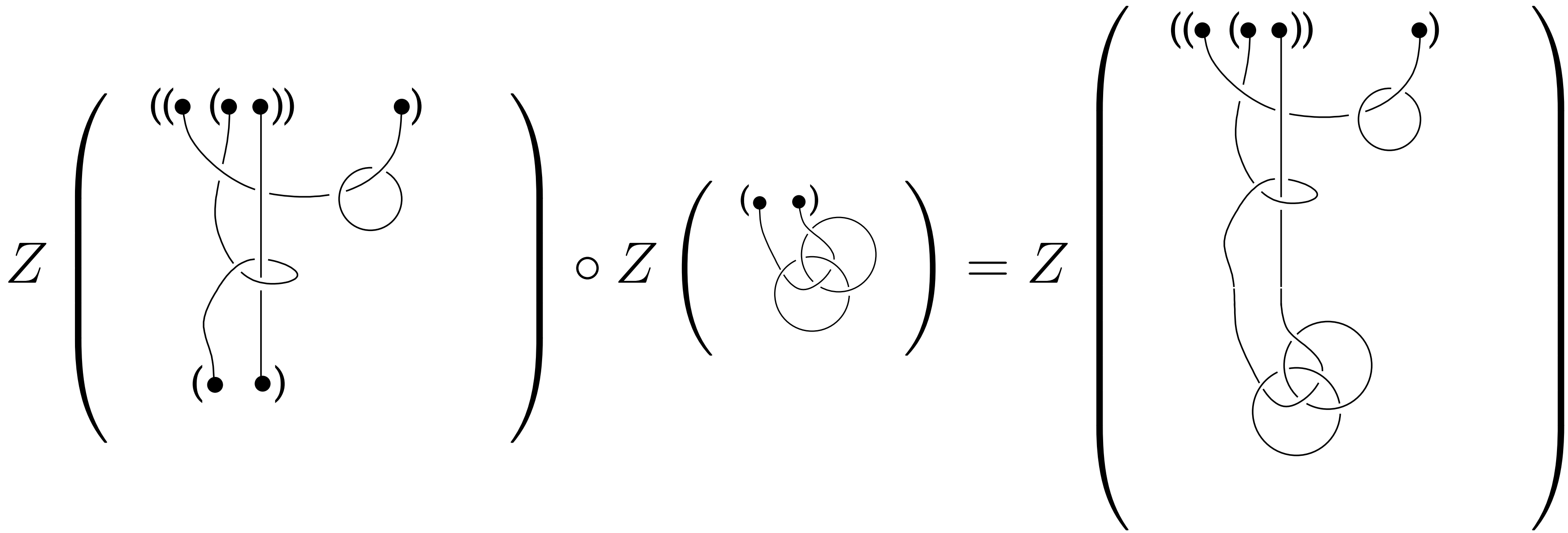
$$Z\,(T) \in \mathcal{A}\,(X) = \frac{\langle \text{Jacobi diagrams } \Gamma \text{ on } X\rangle_{\mathbb{Q}}}{AS, IHX, STU}$$





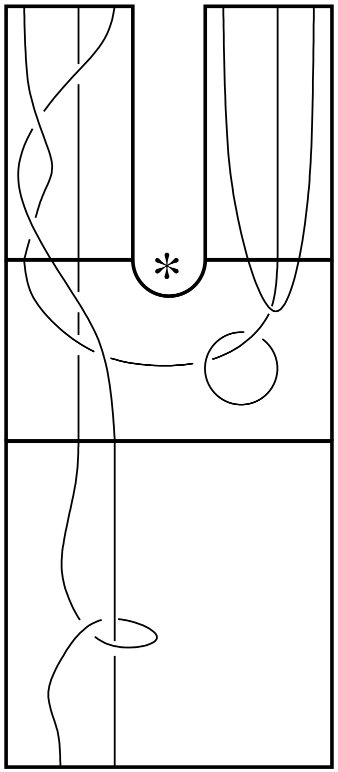
Stacking and Z

The Kontsevich integral respects stacking.



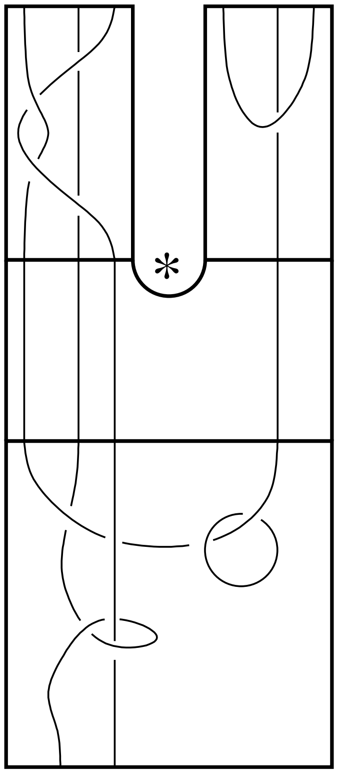
AMR Invariant

Define the (simplified) AMR invariant $Z(L) \in \mathcal{A}(X)$ of a link $L\colon X \hookrightarrow 1_{\Sigma_{g,1}}$ by:



AMR Invariant

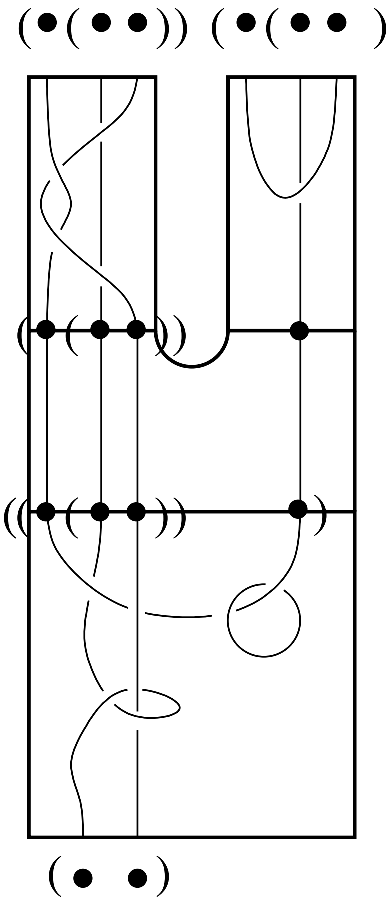
Define the (simplified) AMR invariant $Z(L) \in \mathcal{A}(X)$ of a link $L\colon X \hookrightarrow 1_{\Sigma_{g,1}}$ by:

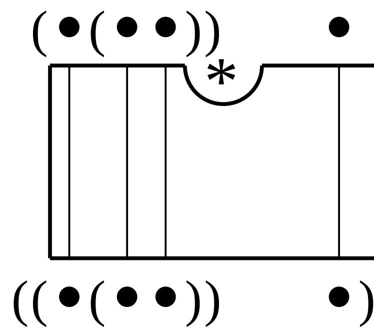


- Isotoping L into position with respect to the AMR decomposition,

AMR Invariant

Define the (simplified) AMR invariant $Z(L) \in \mathcal{A}(X)$ of a link $L\colon X \hookrightarrow 1_{\Sigma_{g,1}}$ by:

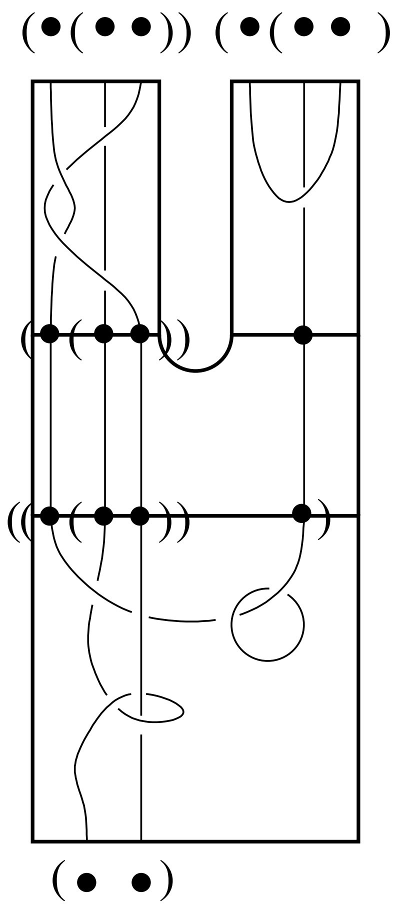


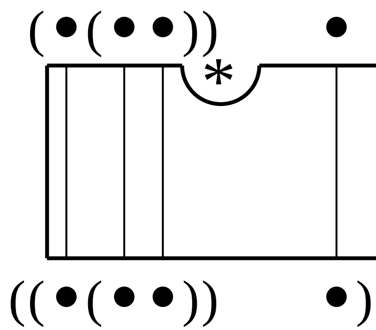
- Isotoping L into position with respect to the AMR decomposition,
- Giving a bracketing consistent with the hexagons: 

,

AMR Invariant

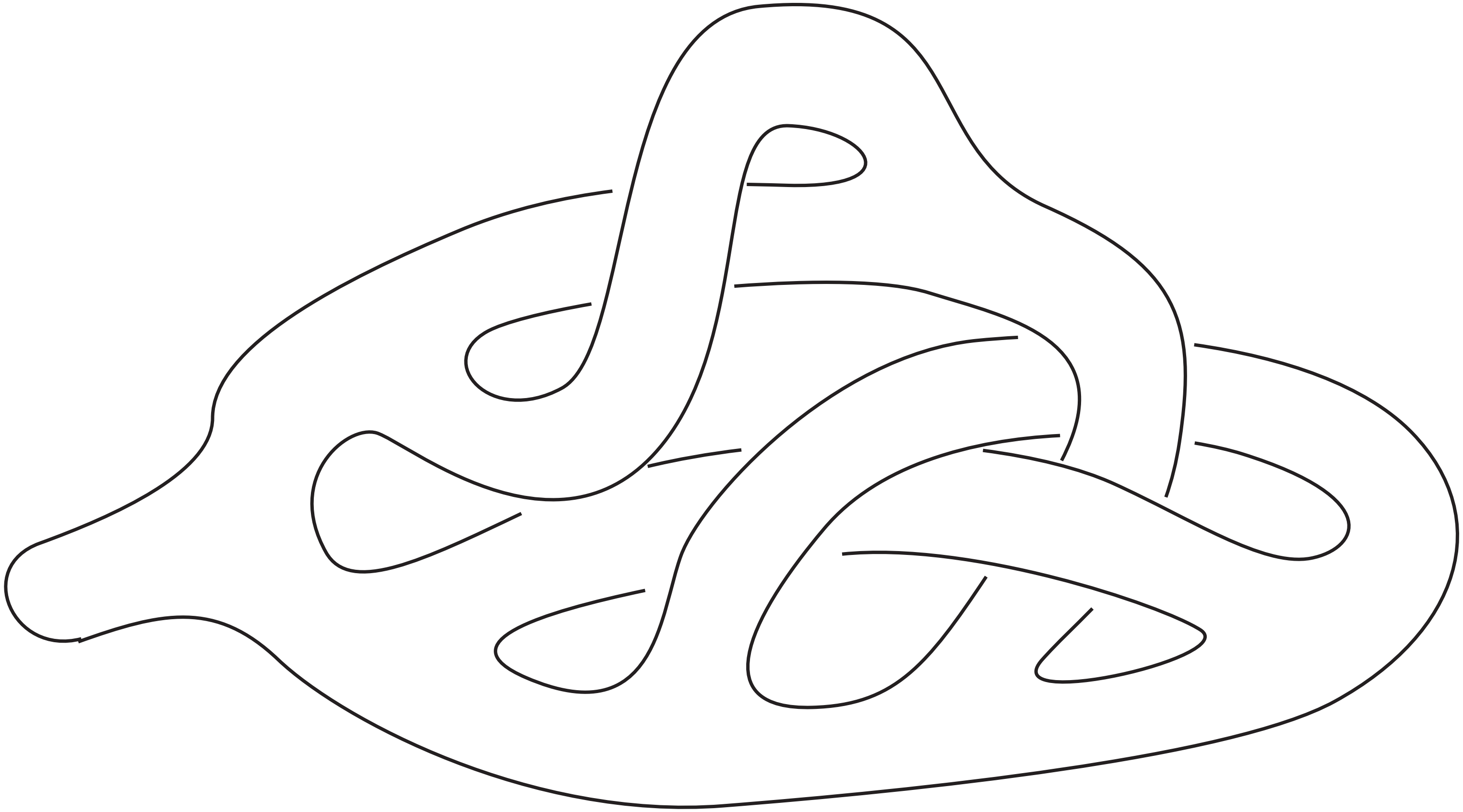
Define the (simplified) AMR invariant $Z(L) \in \mathcal{A}(X)$ of a link $L\colon X \hookrightarrow 1_{\Sigma_{g,1}}$ by:



- Isotoping L into position with respect to the AMR decomposition,
- Giving a bracketing consistent with the hexagons:  ,
- Computing the Kontsevich integral of each square and bi-gon *only*, then stacking them.

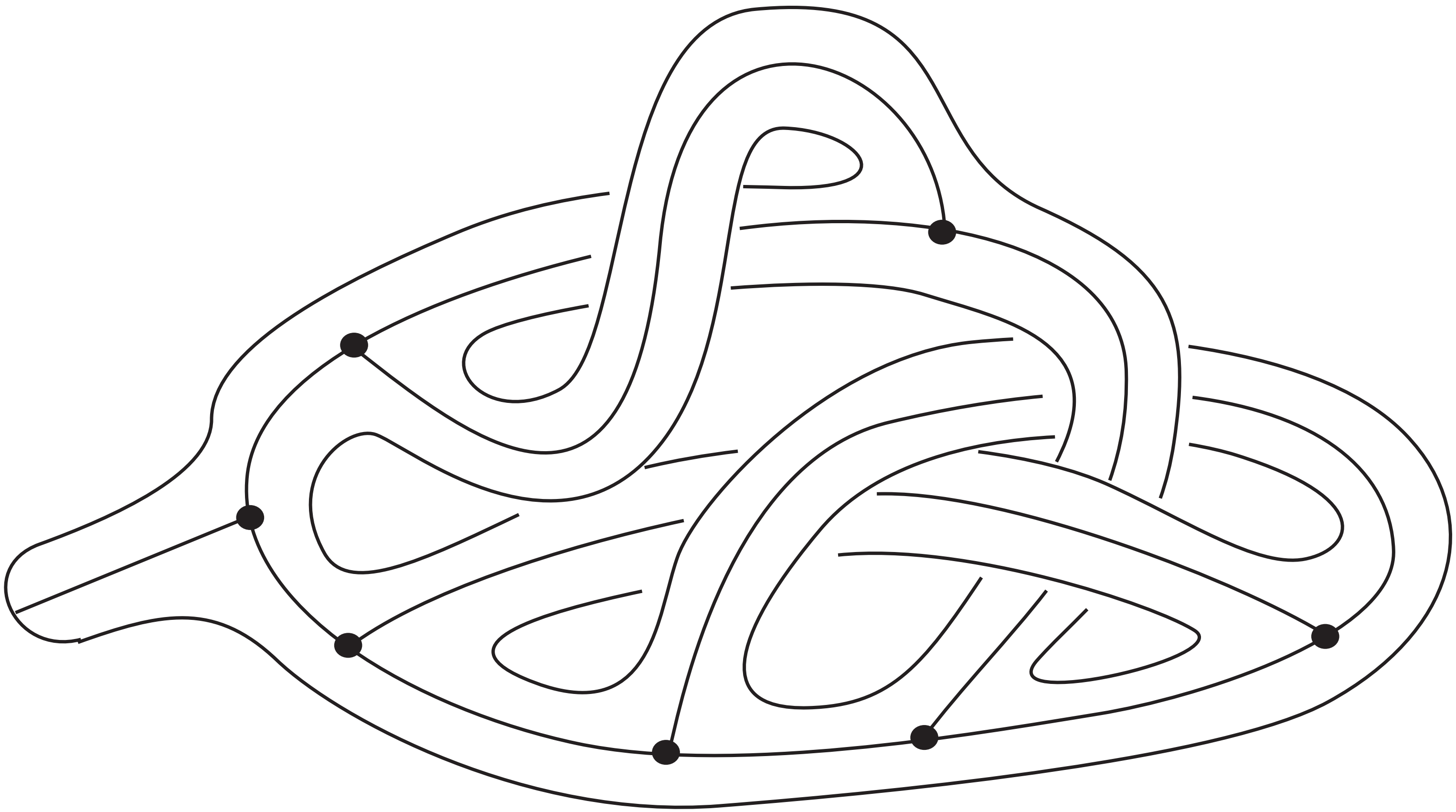
AMR Structure for an arbitrary fatgraph

Let $G \hookrightarrow \Sigma_{g,1}$ be any marked bordered fatgraph. Then, this determines an AMR decomposition of $\Sigma_{g,1}$:



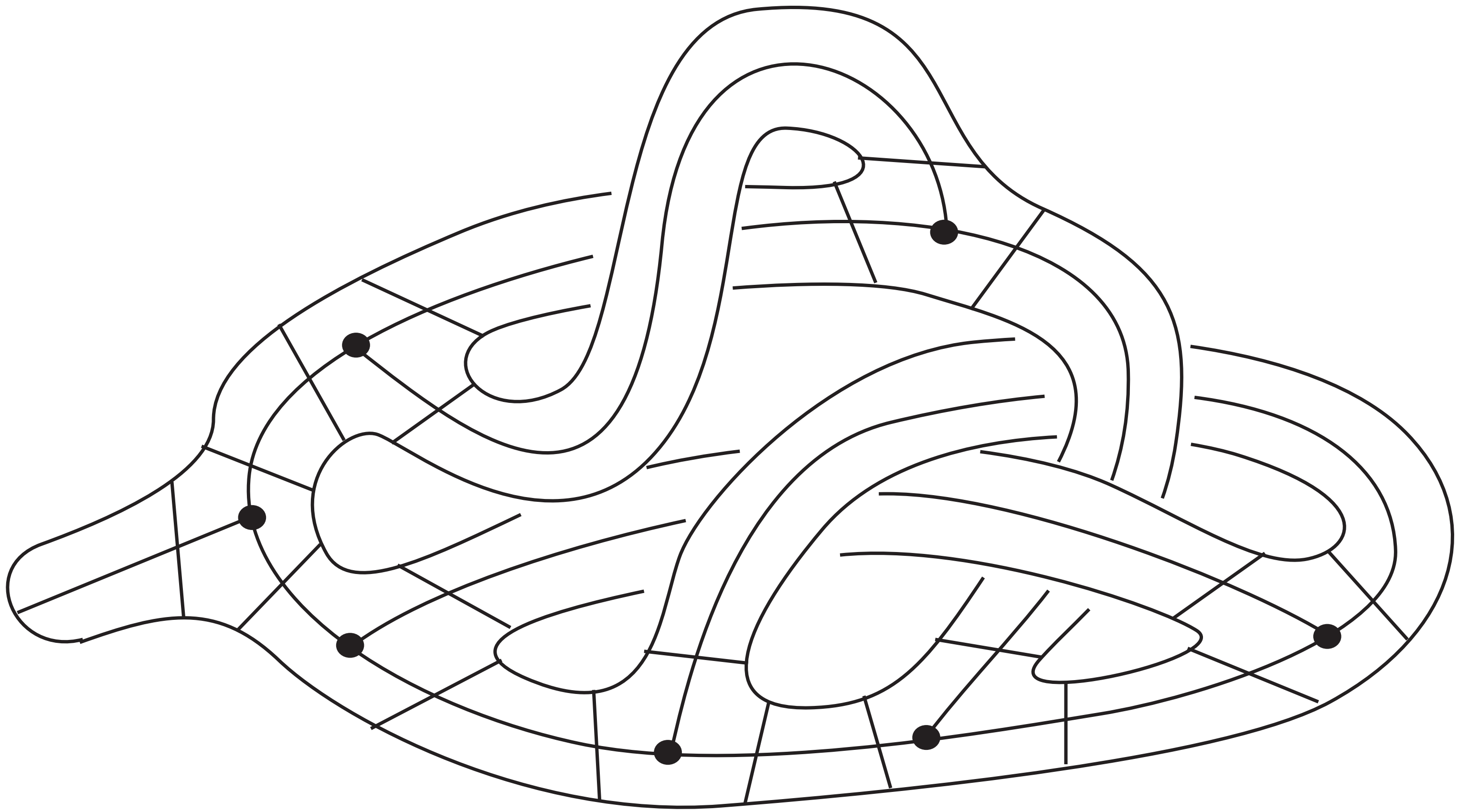
AMR Structure for an arbitrary fatgraph

Let $G \hookrightarrow \Sigma_{g,1}$ be any marked bordered fatgraph. Then, this determines an AMR decomposition of $\Sigma_{g,1}$:



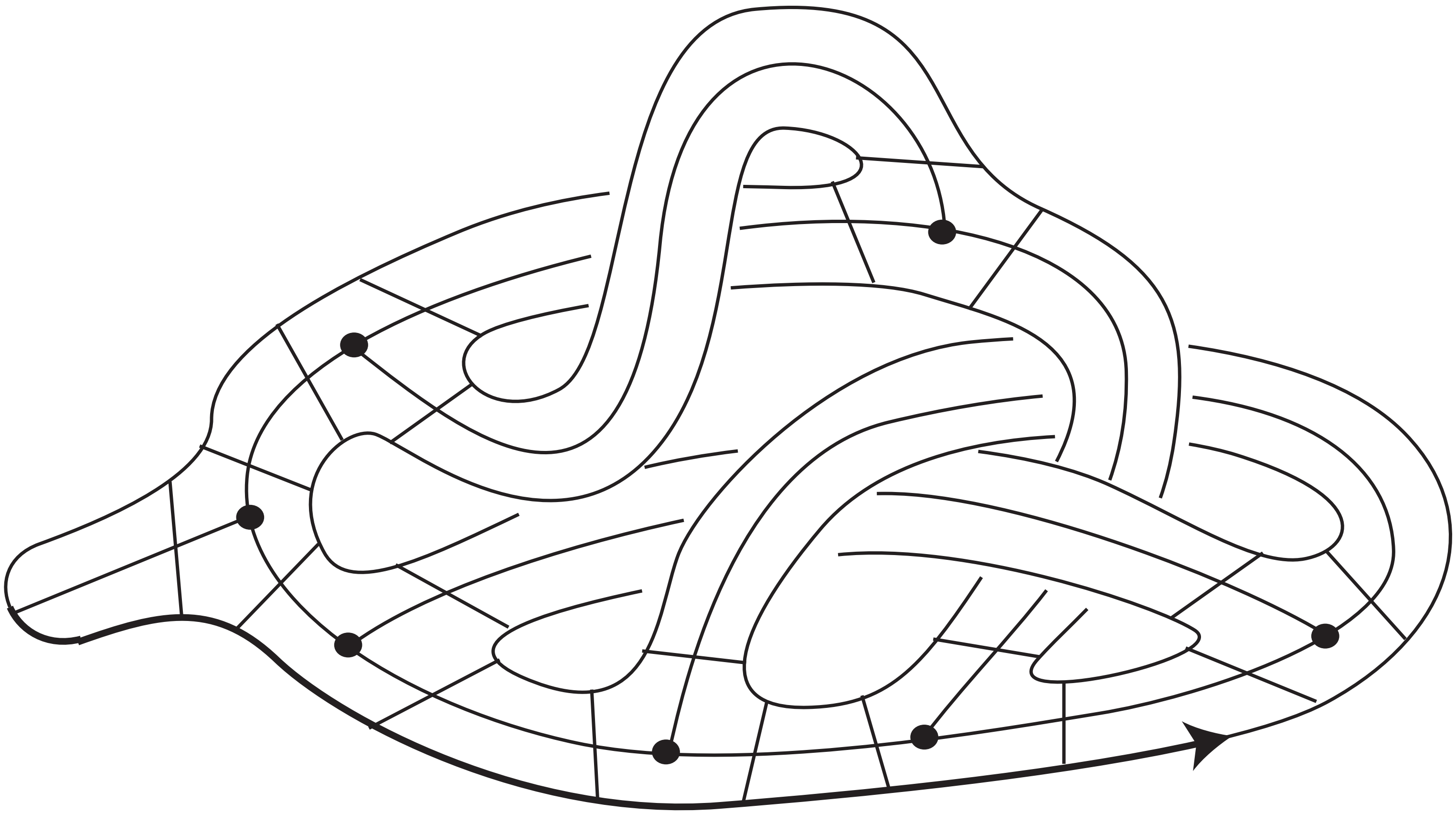
AMR Structure for an arbitrary fatgraph

Let $G \hookrightarrow \Sigma_{g,1}$ be any marked bordered fatgraph. Then, this determines an AMR decomposition of $\Sigma_{g,1}$:



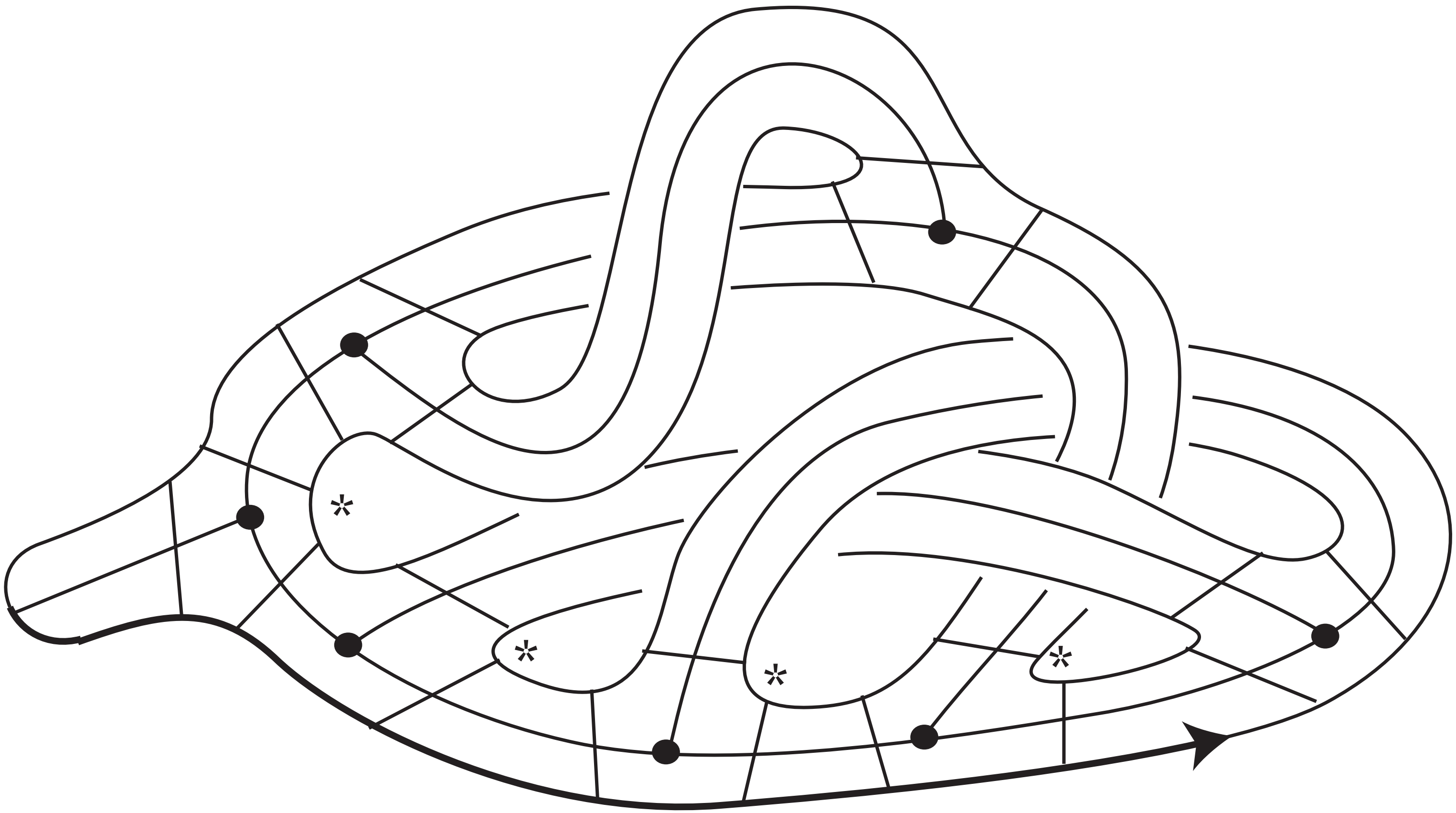
AMR Structure for an arbitrary fatgraph

Let $G \hookrightarrow \Sigma_{g,1}$ be any marked bordered fatgraph. Then, this determines an AMR decomposition of $\Sigma_{g,1}$:



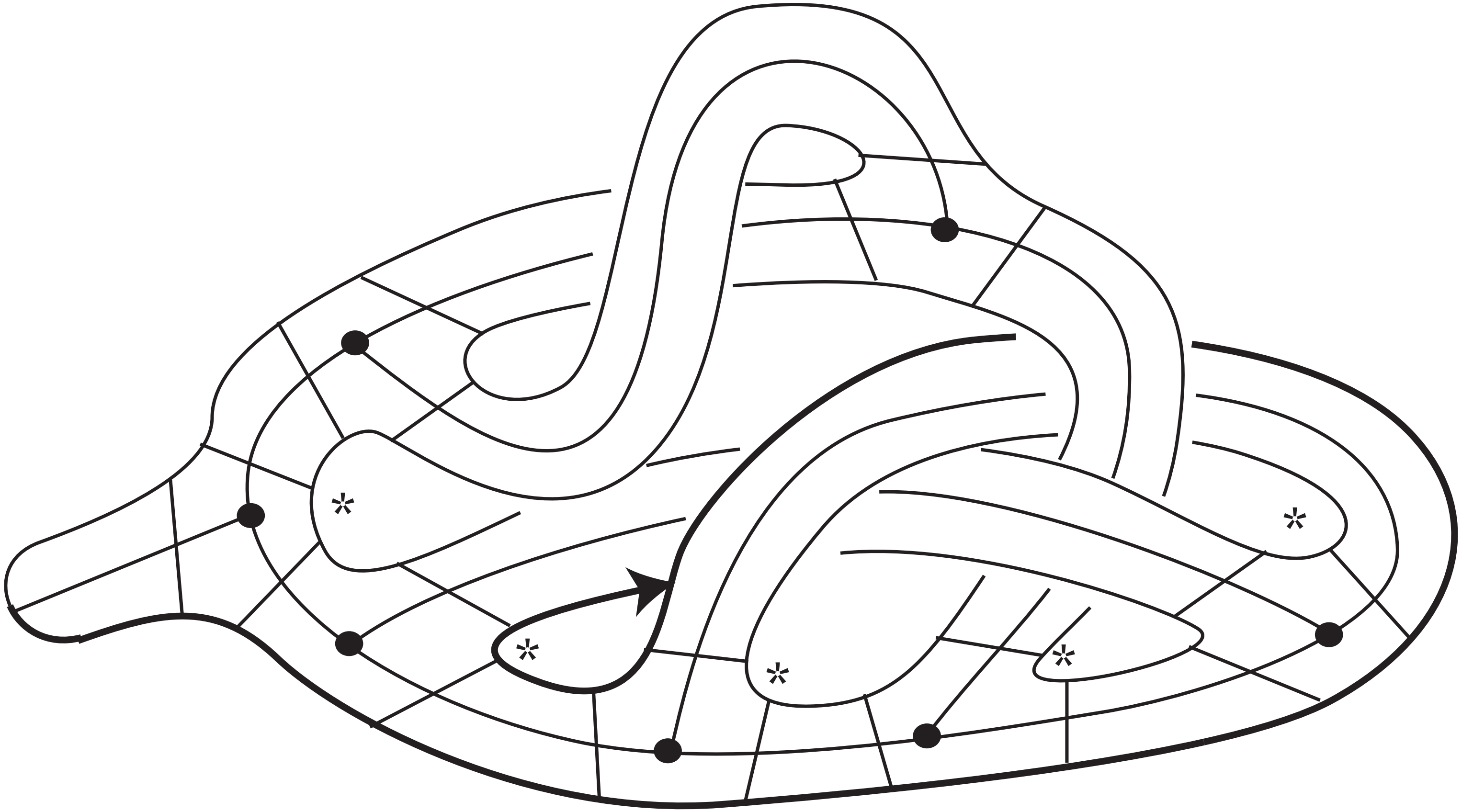
AMR Structure for an arbitrary fatgraph

Let $G \hookrightarrow \Sigma_{g,1}$ be any marked bordered fatgraph. Then, this determines an AMR decomposition of $\Sigma_{g,1}$:



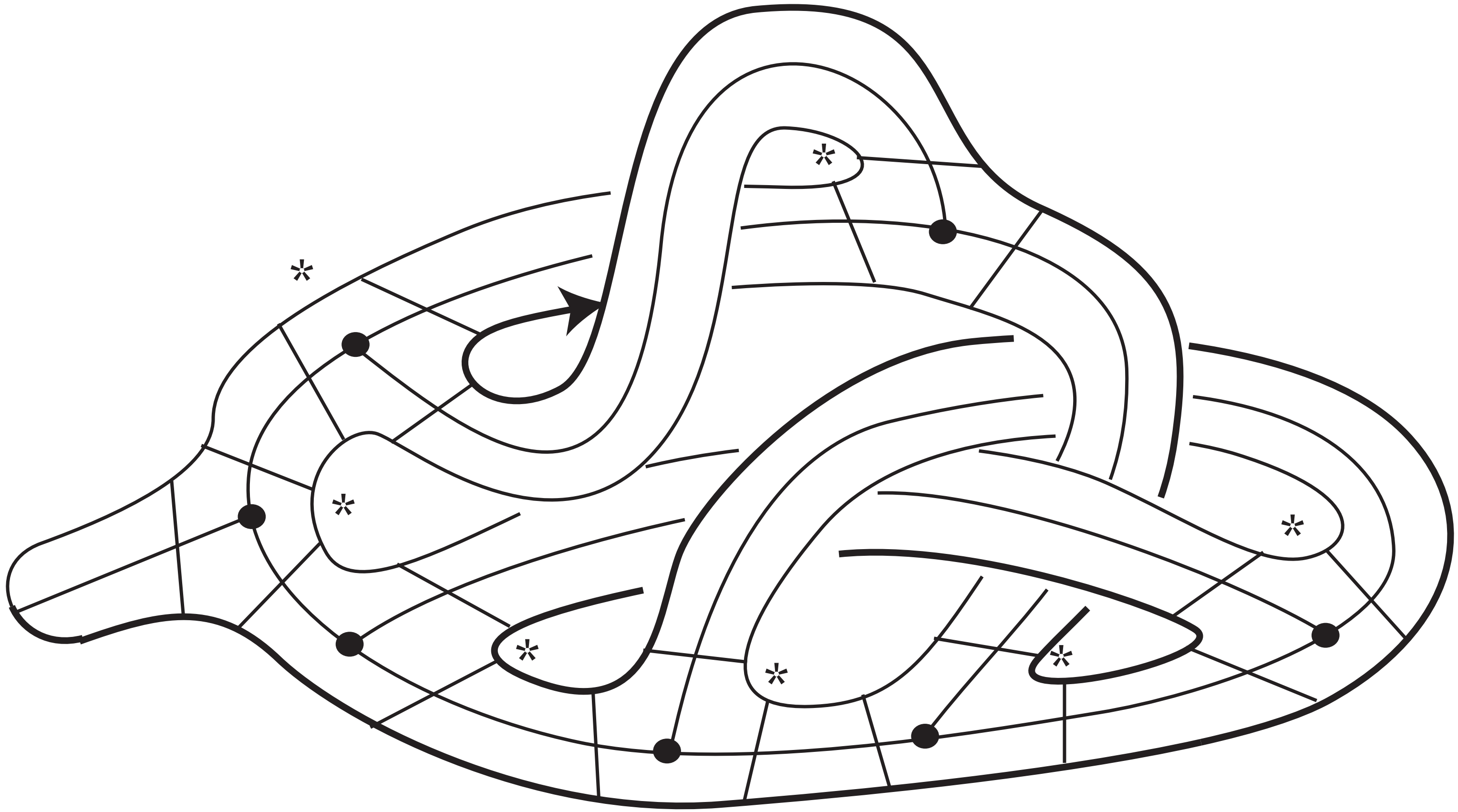
AMR Structure for an arbitrary fatgraph

Let $G \hookrightarrow \Sigma_{g,1}$ be any marked bordered fatgraph. Then, this determines an AMR decomposition of $\Sigma_{g,1}$:



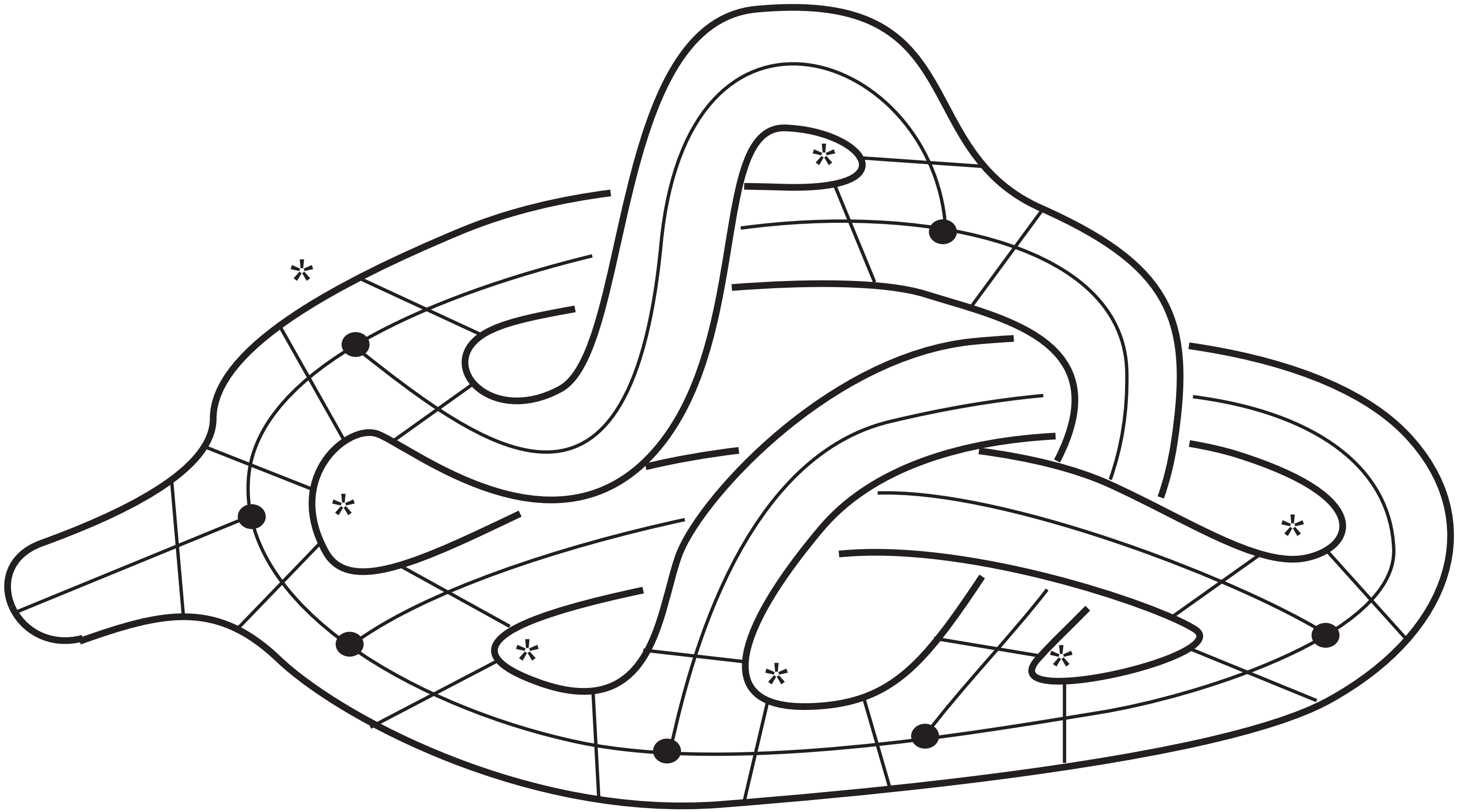
AMR Structure for an arbitrary fatgraph

Let $G \hookrightarrow \Sigma_{g,1}$ be any marked bordered fatgraph. Then, this determines an AMR decomposition of $\Sigma_{g,1}$:



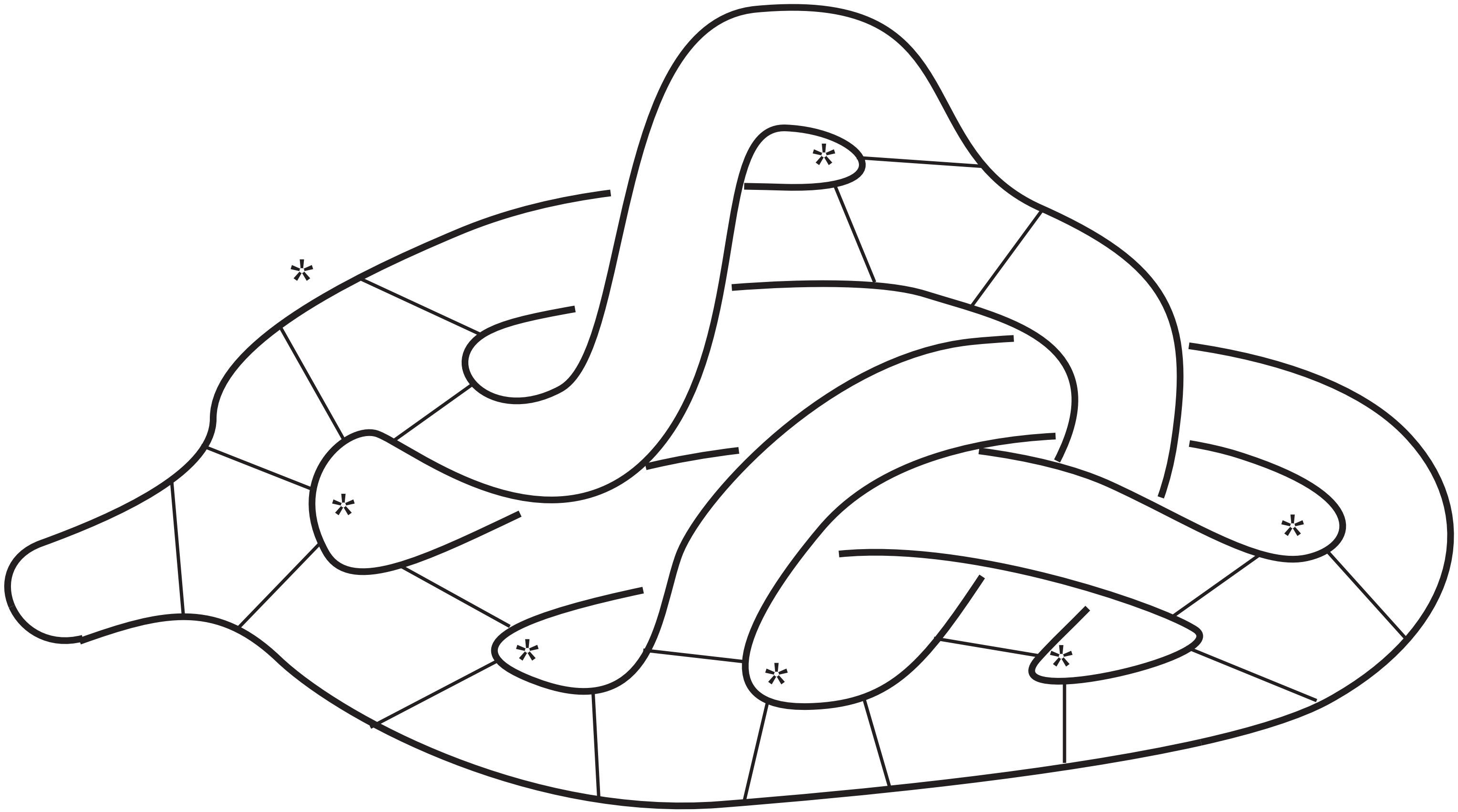
AMR Structure for an arbitrary fatgraph

Let $G \hookrightarrow \Sigma_{g,1}$ be any marked bordered fatgraph. Then, this determines an AMR decomposition of $\Sigma_{g,1}$:



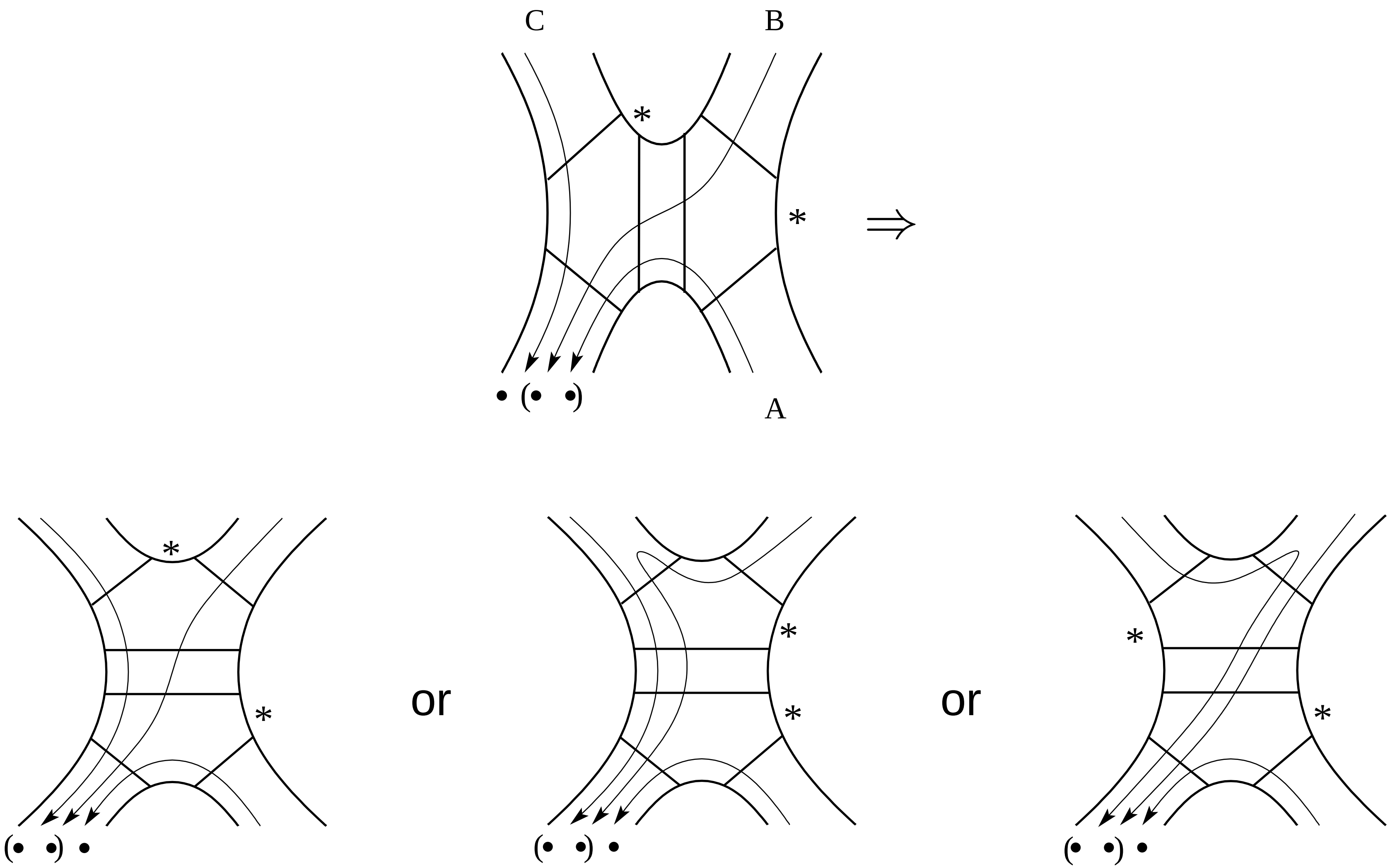
AMR Structure for an arbitrary fatgraph

Let $G \hookrightarrow \Sigma_{g,1}$ be any marked bordered fatgraph. Then, this determines an AMR decomposition of $\Sigma_{g,1}$:



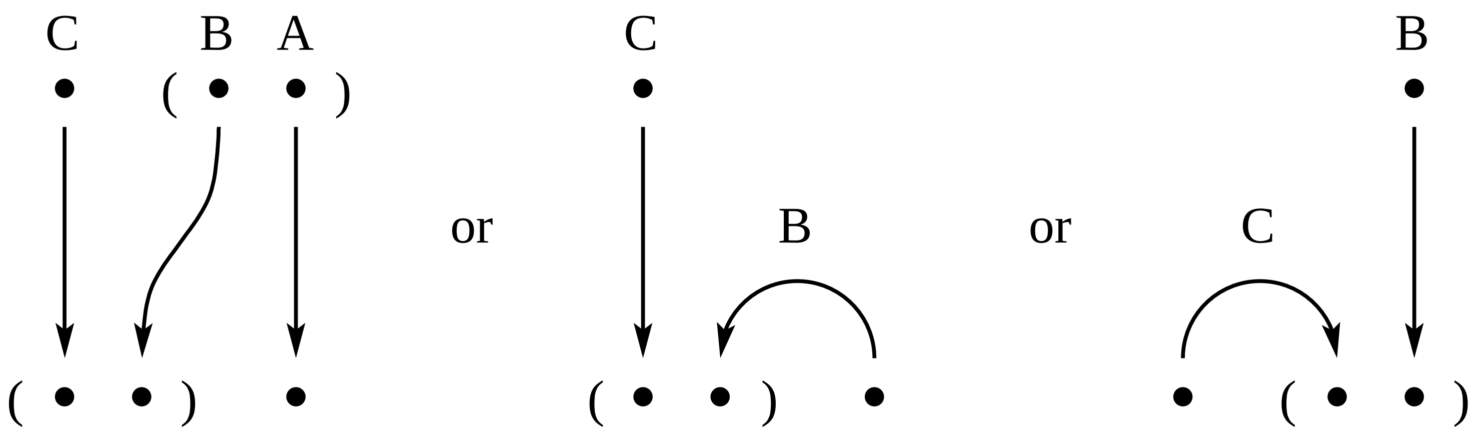
AMR and Whitehead moves

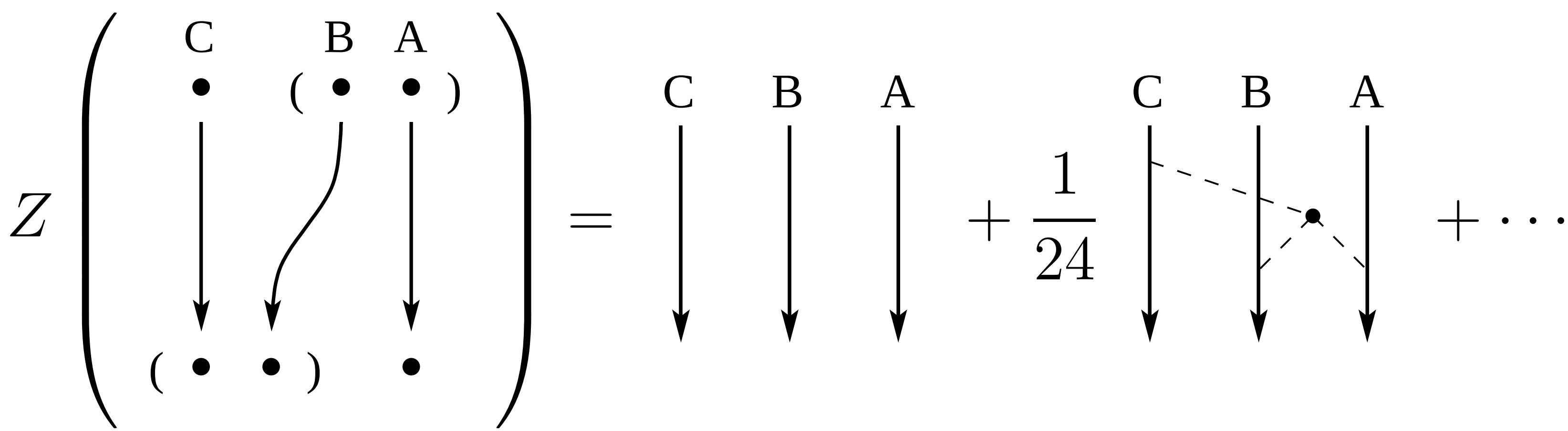
A Whitehead move affects the AMR invariant:



AMR and Whitehead moves

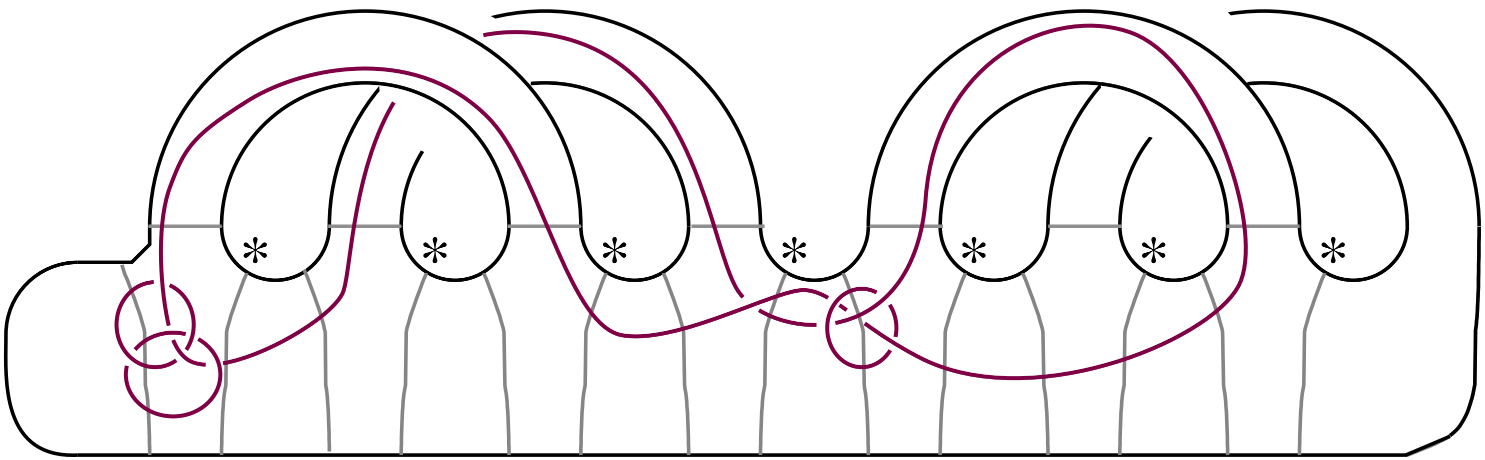
The changes are given by one of the three possibilities:





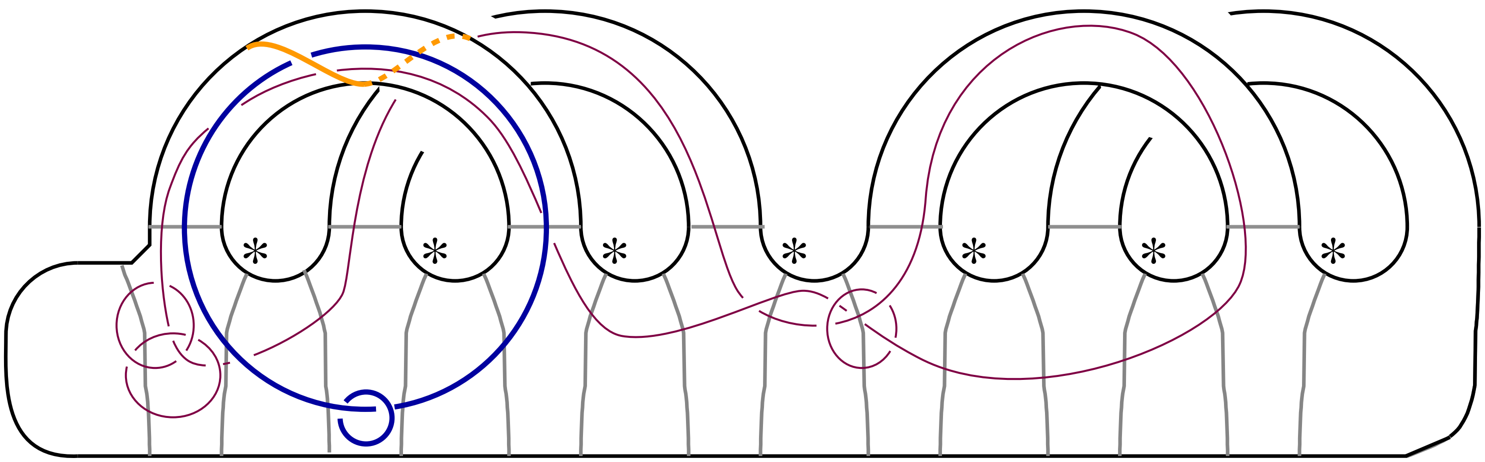
∇_G Invariant

By adding a couple of ingredients,



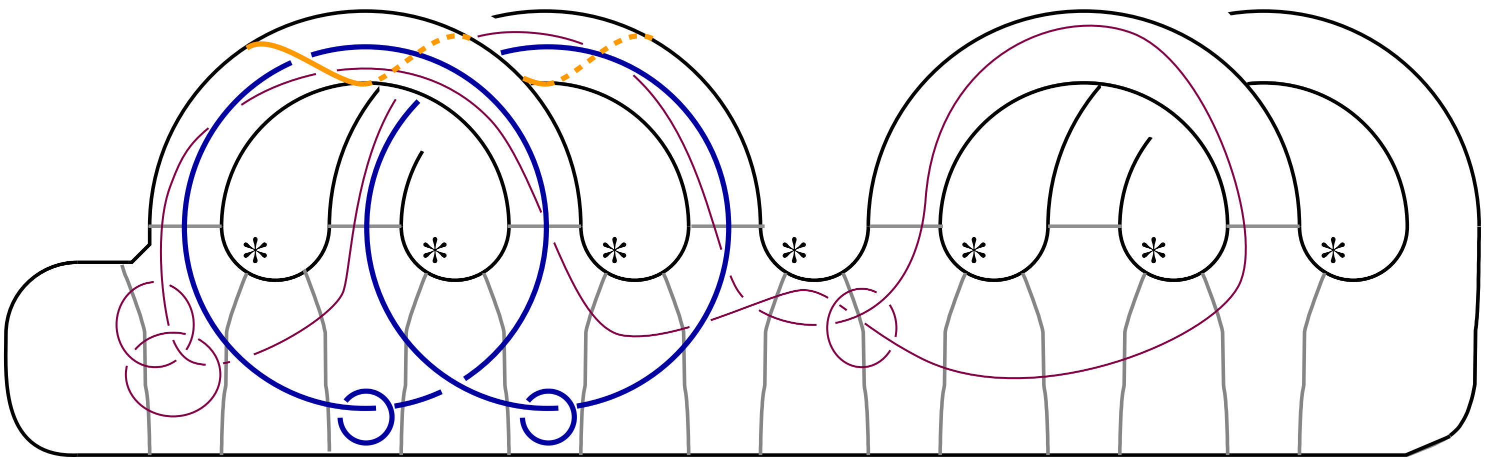
∇_G Invariant

By adding a couple of ingredients,



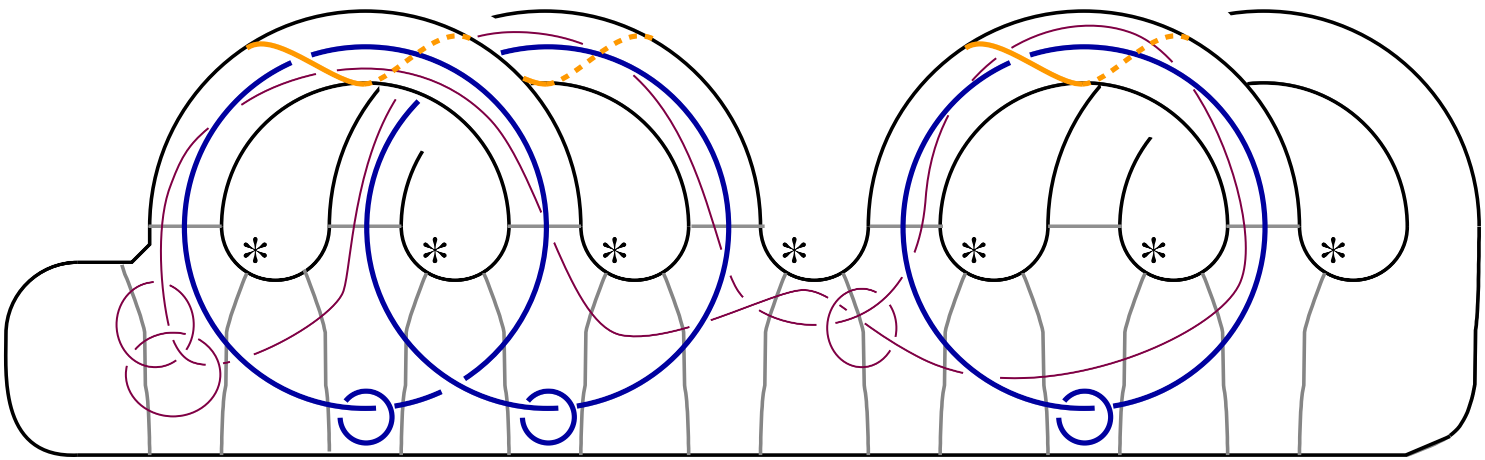
∇_G Invariant

By adding a couple of ingredients,



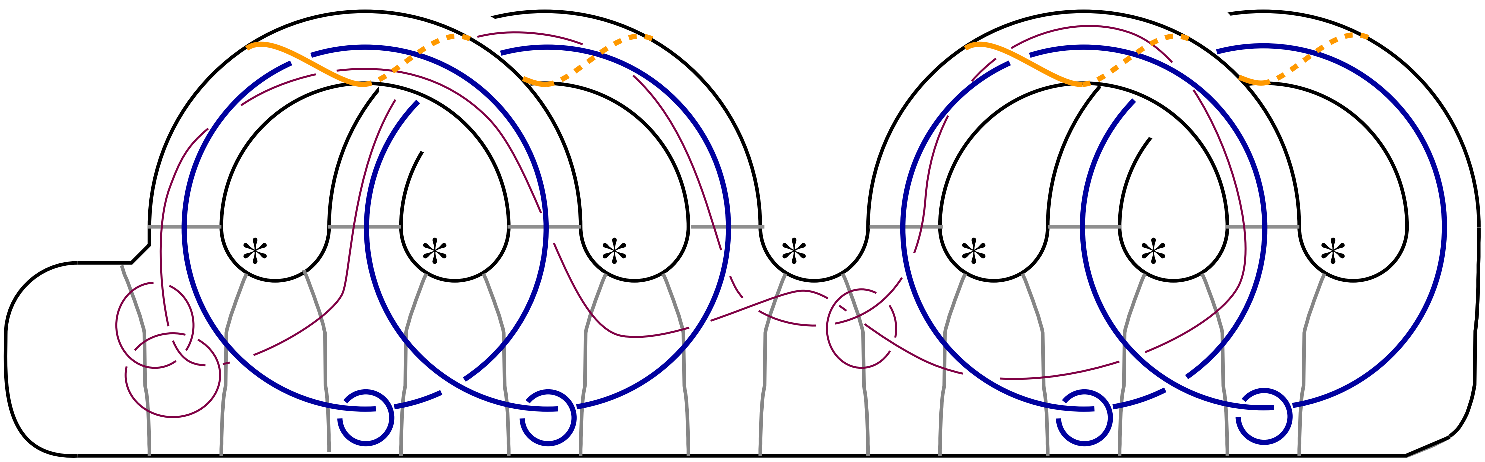
∇_G Invariant

By adding a couple of ingredients,



∇_G Invariant

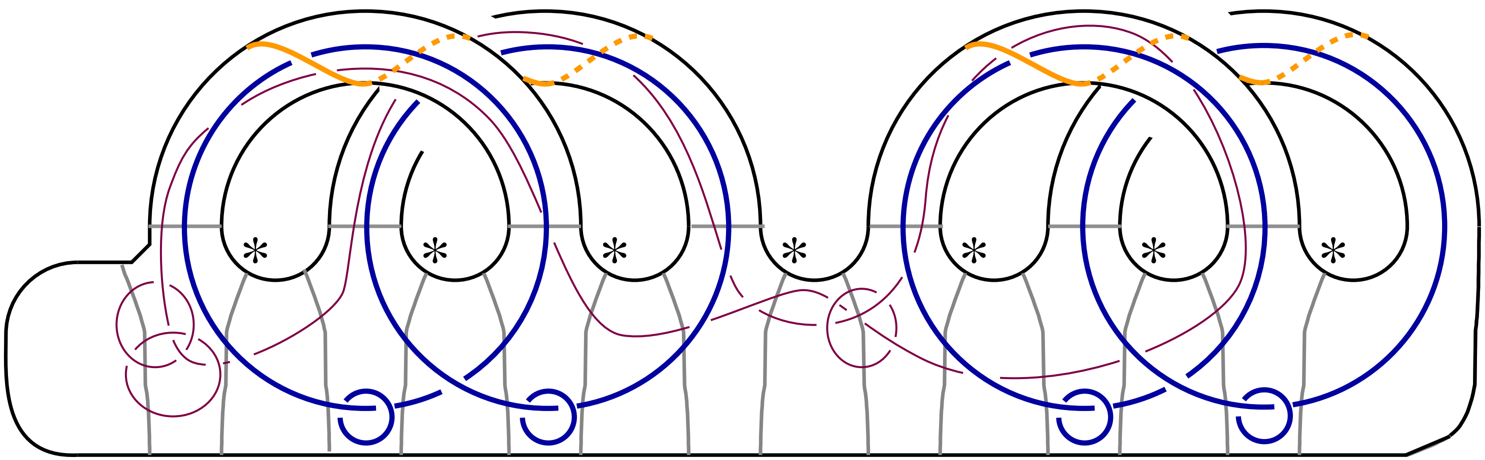
By adding a couple of ingredients, we can modify the AMR invariant to obtain an invariant ∇_G, with image $\mathcal{A}(\uparrow^{2g})$, of surgery cylinders over $\Sigma_{g,1}$ (3-manifolds obtained by surgery on links in $1_{\Sigma_{g,1}}$). In fact,



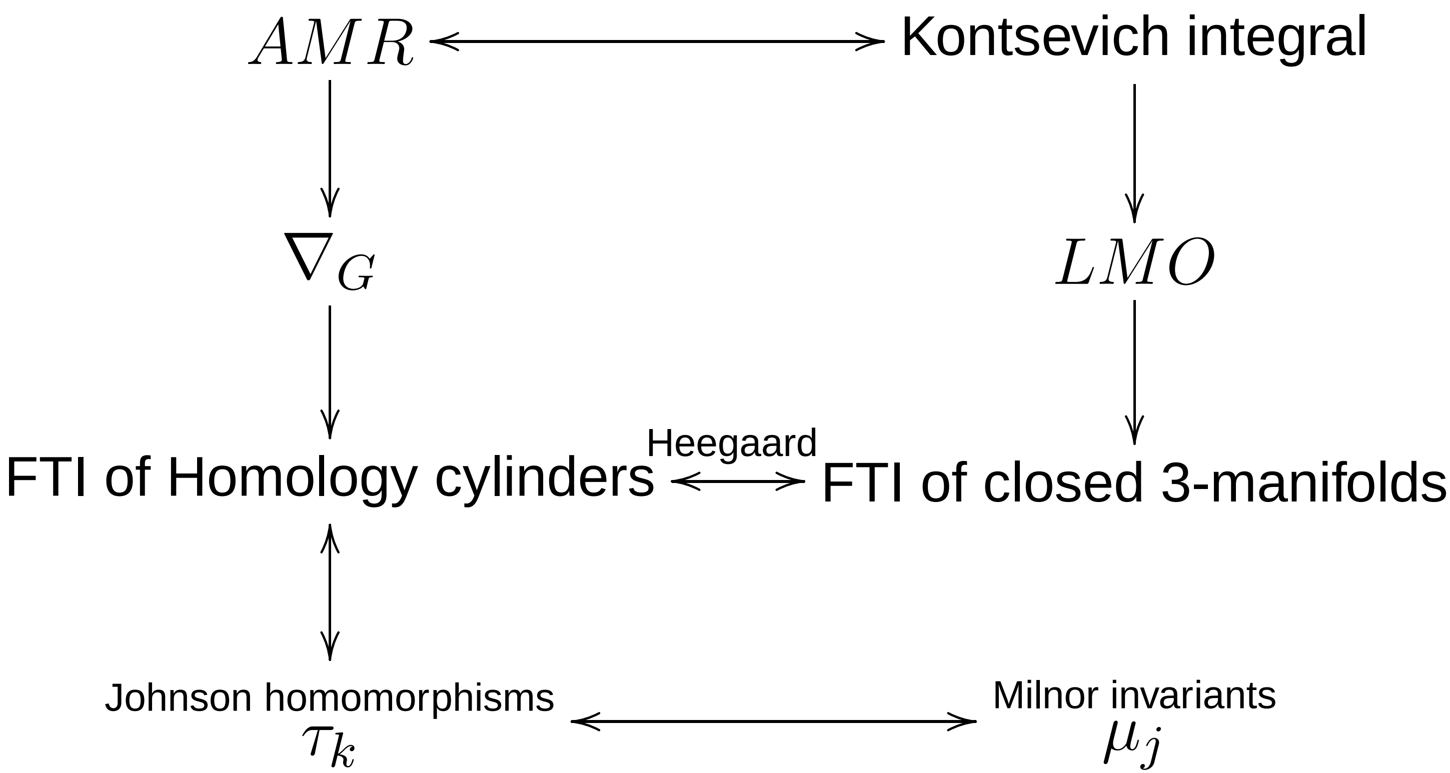
∇_G Invariant

By adding a couple of ingredients, we can modify the AMR invariant to obtain an invariant ∇_G, with image $\mathcal{A}(\uparrow^{2g})$, of surgery cylinders over $\Sigma_{g,1}$ (3-manifolds obtained by surgery on links in $1_{\Sigma_{g,1}}$). In fact,

Theorem (ABMP)**.** *For any marked bordered fatgraph $G \hookrightarrow \Sigma_{g,1}$, the invariant ∇_G is a universal finite type invariant of homology cylinders over $\Sigma_{g,1}$.*



Connections



The First Johnson Homomorphism

Let $\mathcal{I}_{g,1}$ be the Torelli subgroup of $MC_{g,1}$:
the subgroup which acts trivially on H.

$$\tau_1 \colon \mathcal{I}_{g,1} \to \Lambda^3 H \subset H^* \otimes \Lambda^2 H$$

is defined by:

$$\varphi \mapsto \left(X \mapsto \varphi(x)x^{-1} \in \frac{[\pi,\pi]}{[\pi,[\pi,\pi]]} \text{ for } [x] = X, x \in \pi \right)$$

More generally, we have the higher Johnson homomorphisms

$$\tau_k \colon \mathcal{I}_{g,1}[k] \to D_k(H) \subset H^* \otimes \mathcal{L}_{k+1}(H)$$

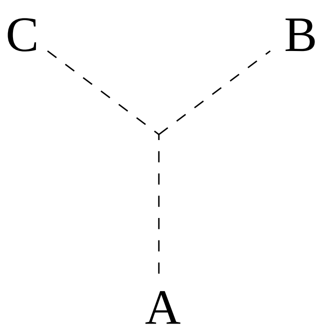
Image of τ_1

The images of the Johnson homomorphisms lie in certain spaces of Jacobi diagrams:

$$D_k(H) \subset D_k(H_{\mathbb{Q}}) = \frac{\left\langle \begin{array}{c}\text{Connected Jacobi “tree” diagrams} \\ \text{w/ } k+2\ H\text{-labelled 1-valent vertices}\end{array} \right\rangle_{\mathbb{Q}}}{AS, IHX, \text{multi-linearity in } H}.$$

In particular,

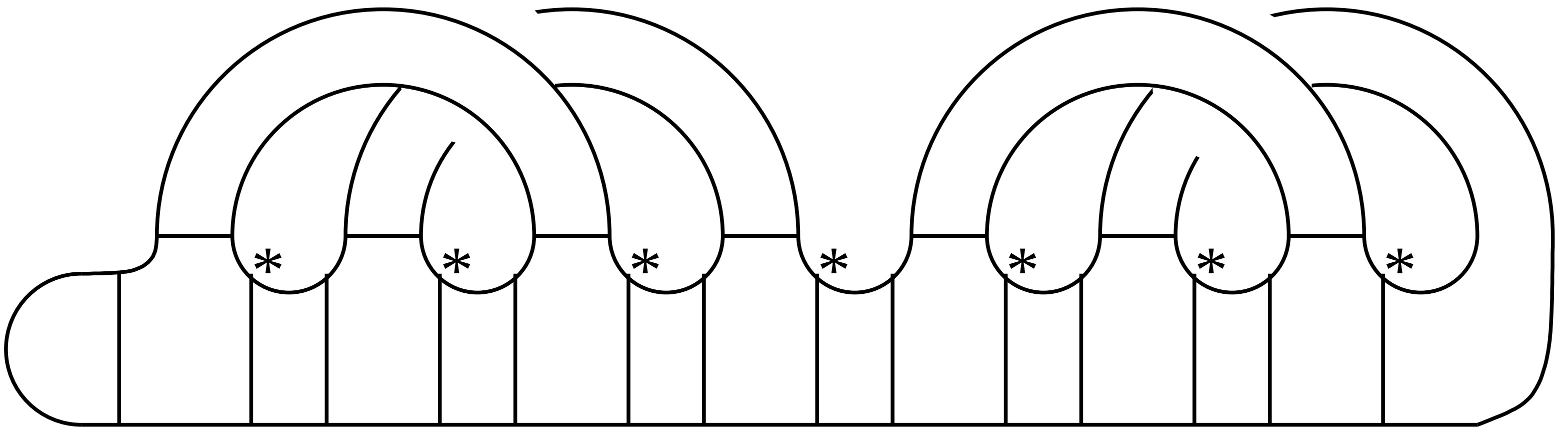
$$\tau_1 : \mathcal{I}_{g,1} \to \Lambda^3 H \cong D_1(H) \cong \frac{\langle Y\text{-graphs labelled by } H \rangle_{\mathbb{Z}}}{AS, \text{multi-linearity in } H}$$



$$\mapsto A \wedge B \wedge C.$$

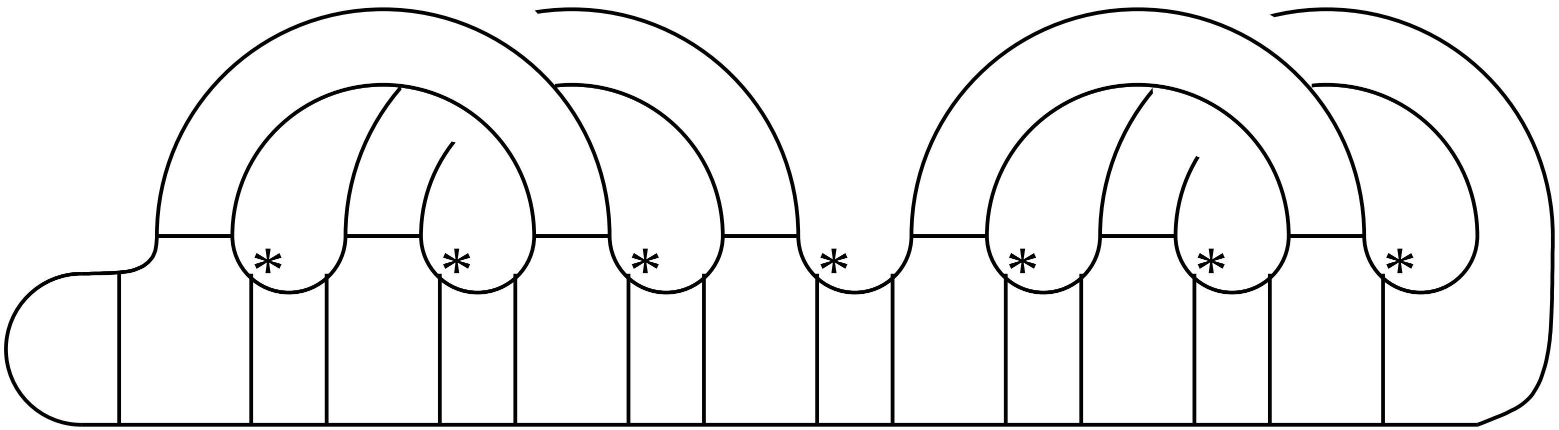
A different perspective

Let’s change perspectives ....



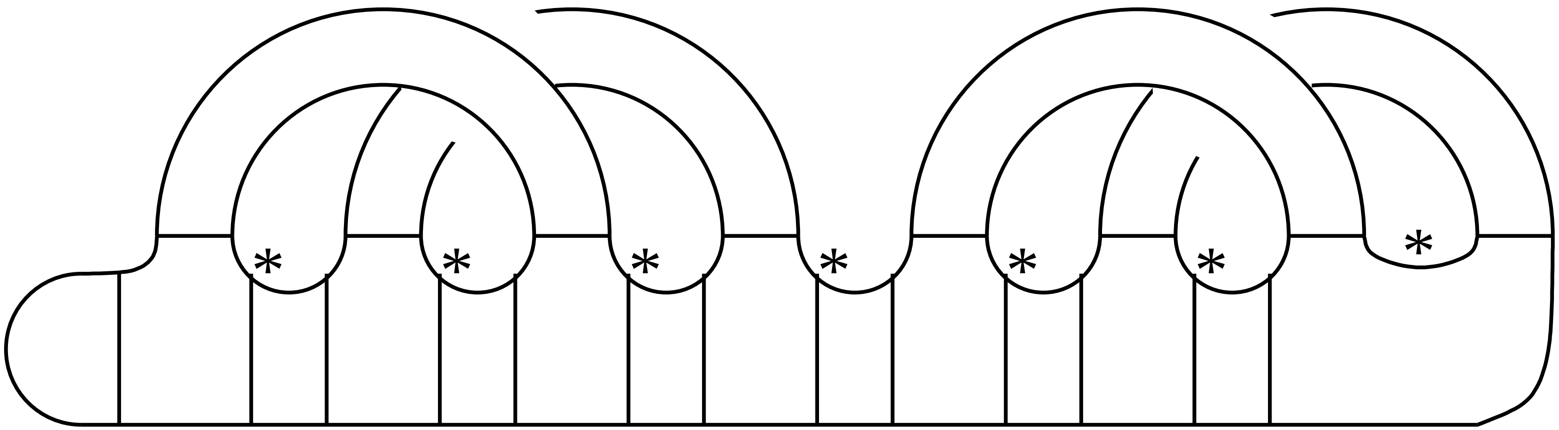
A different perspective

Let's change perspectives ....



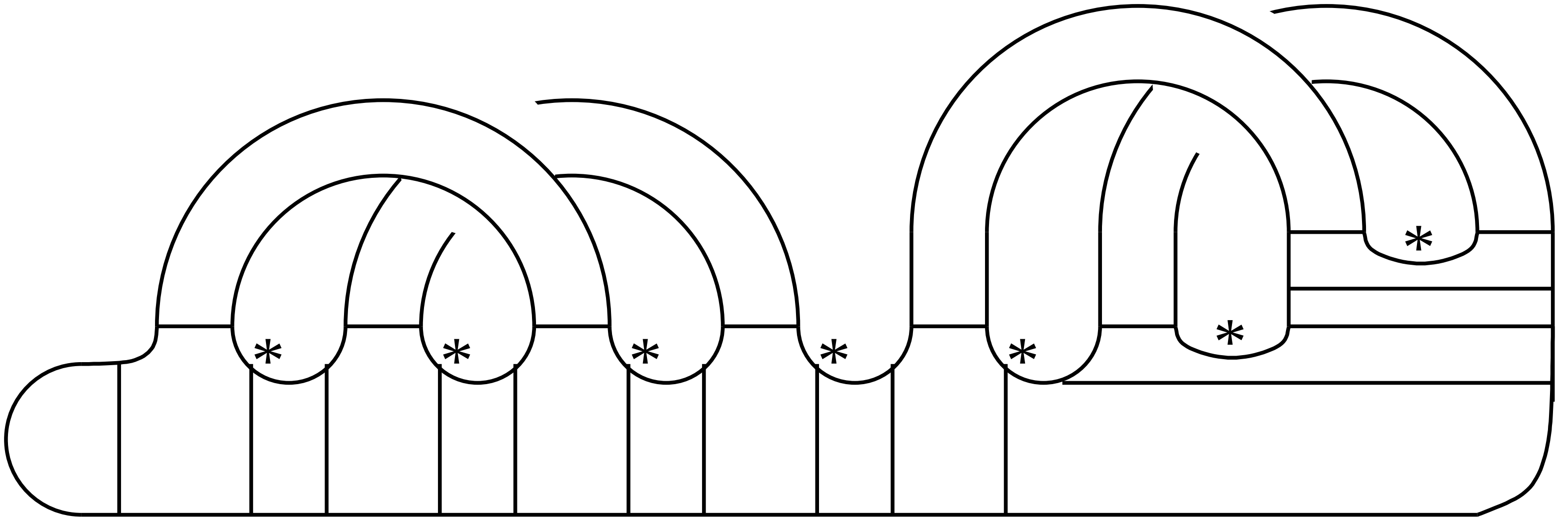
A different perspective

Let’s change perspectives ....



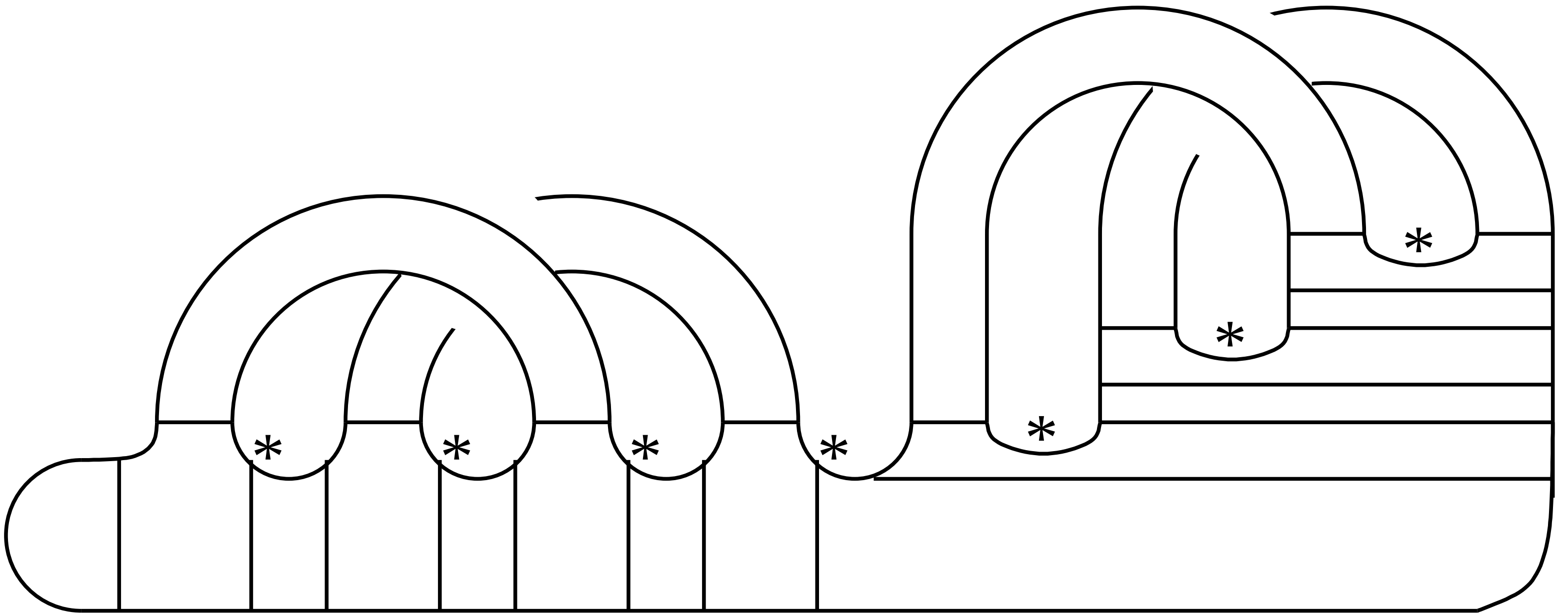
A different perspective

Let's change perspectives ....



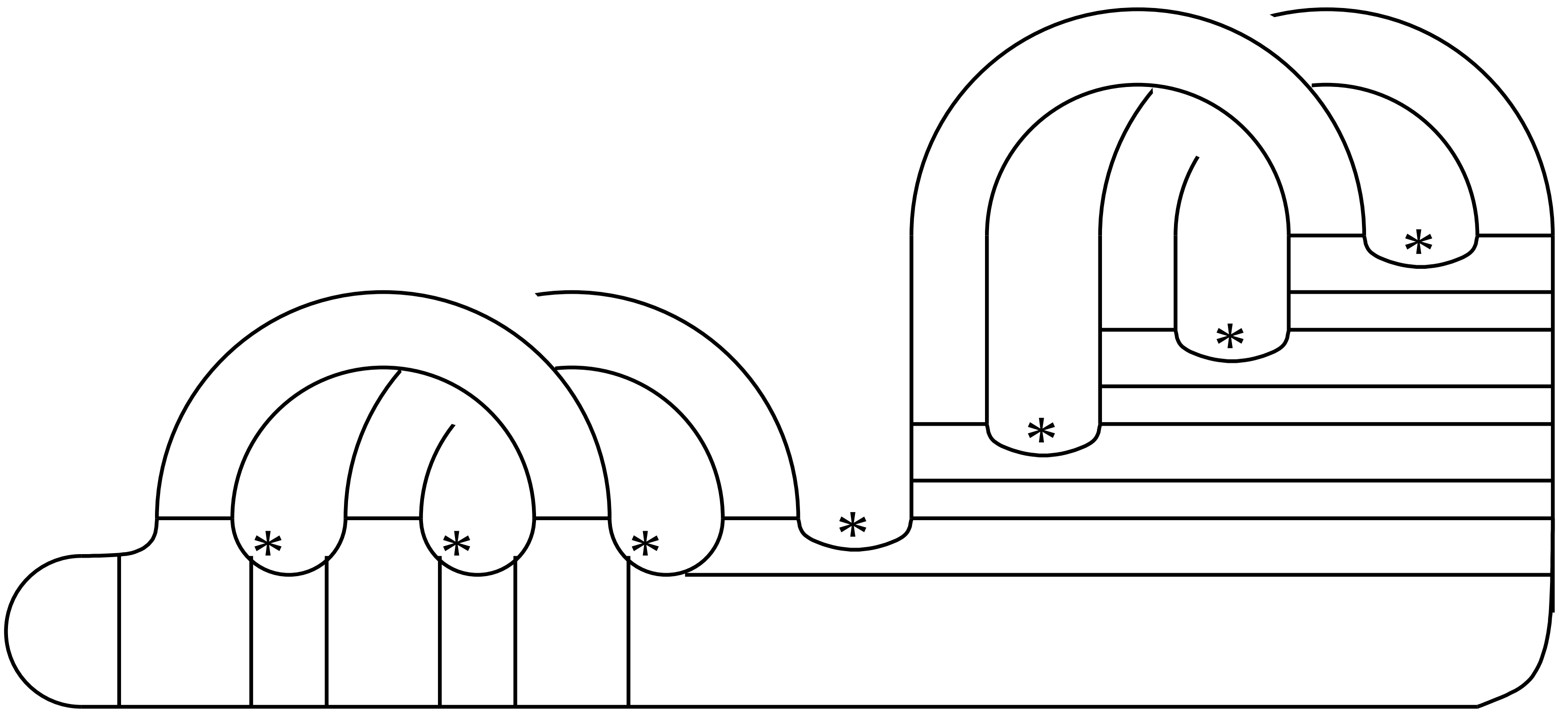
A different perspective

Let’s change perspectives ....



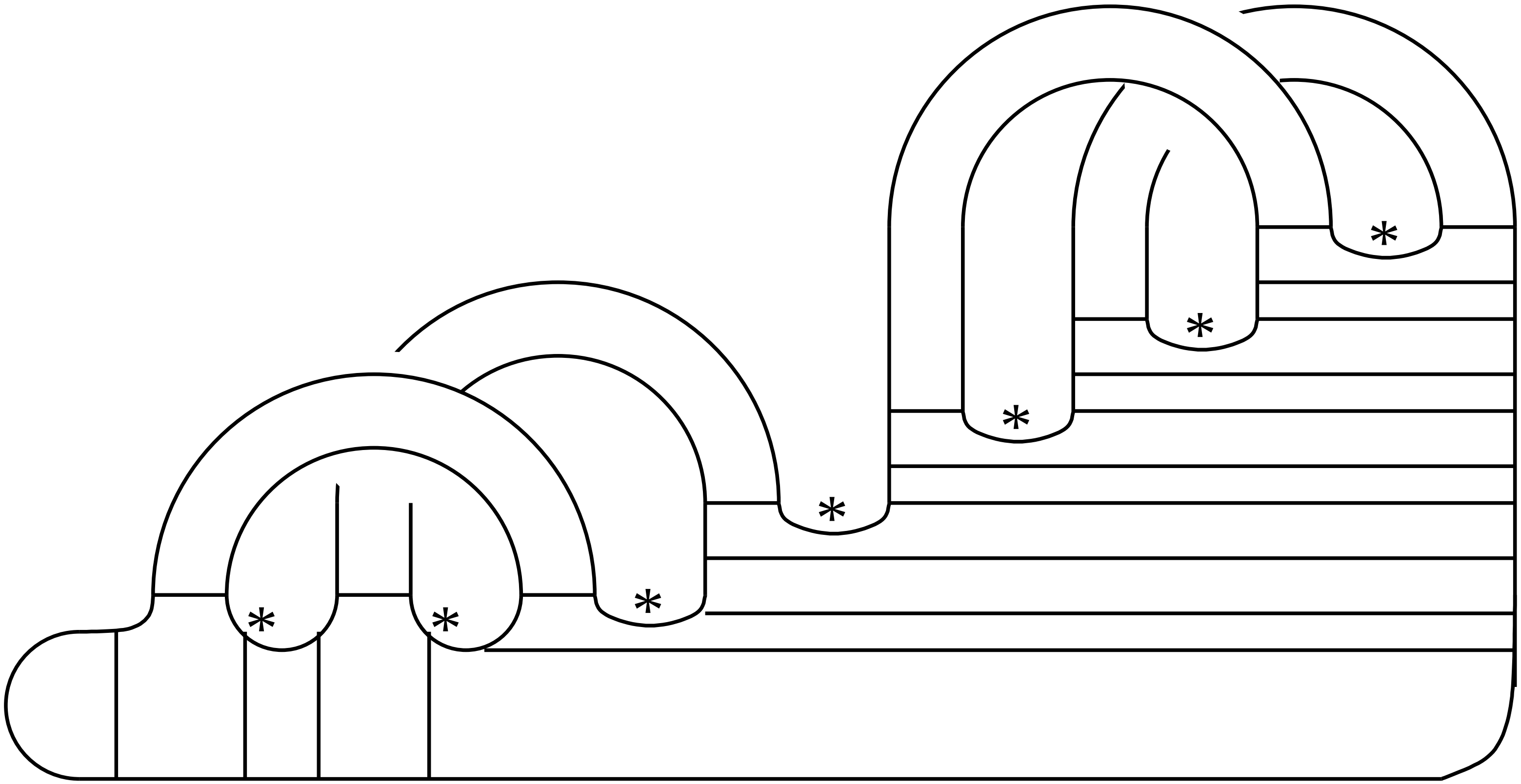
A different perspective

Let's change perspectives ....



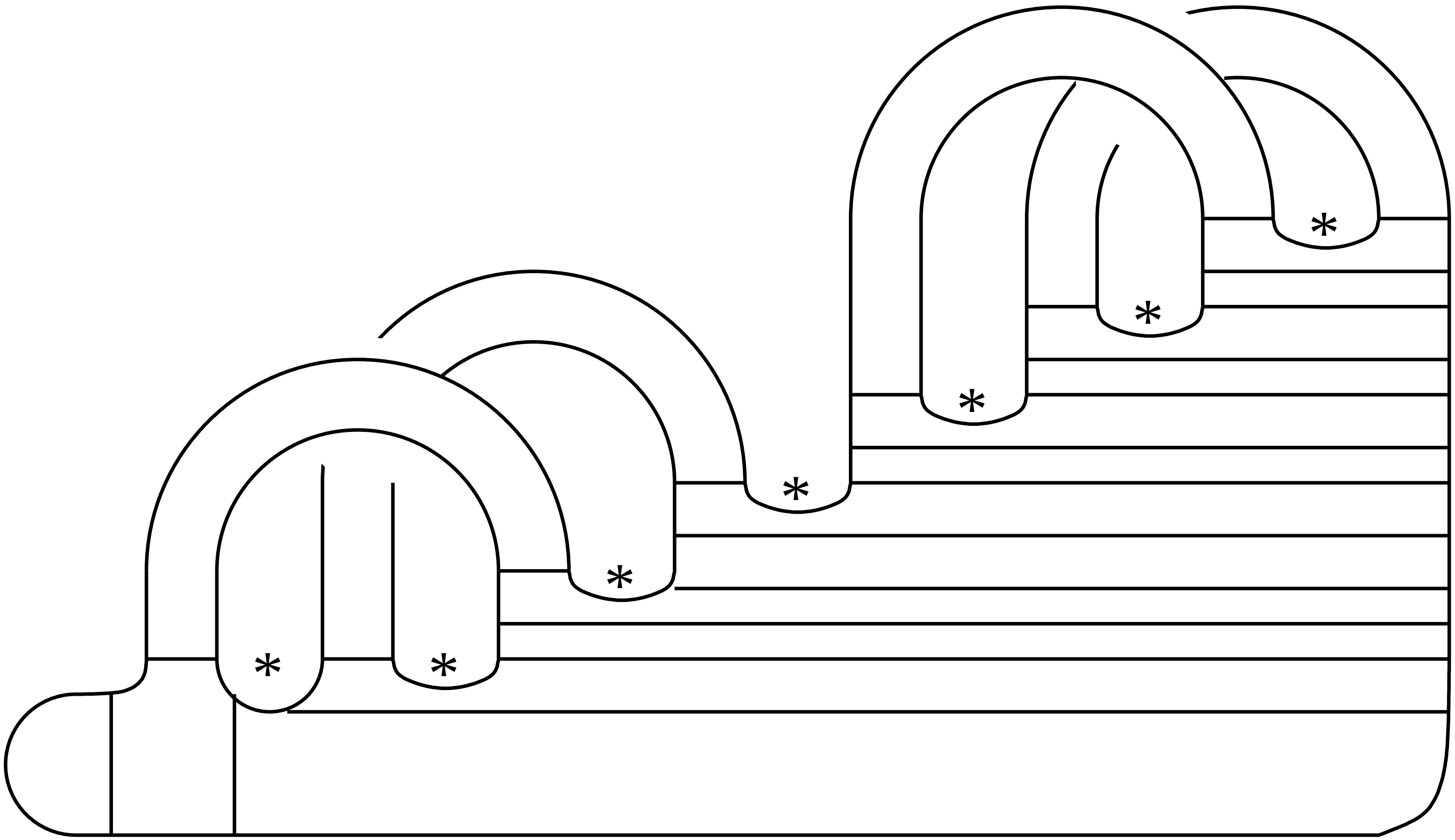
A different perspective

Let's change perspectives ....



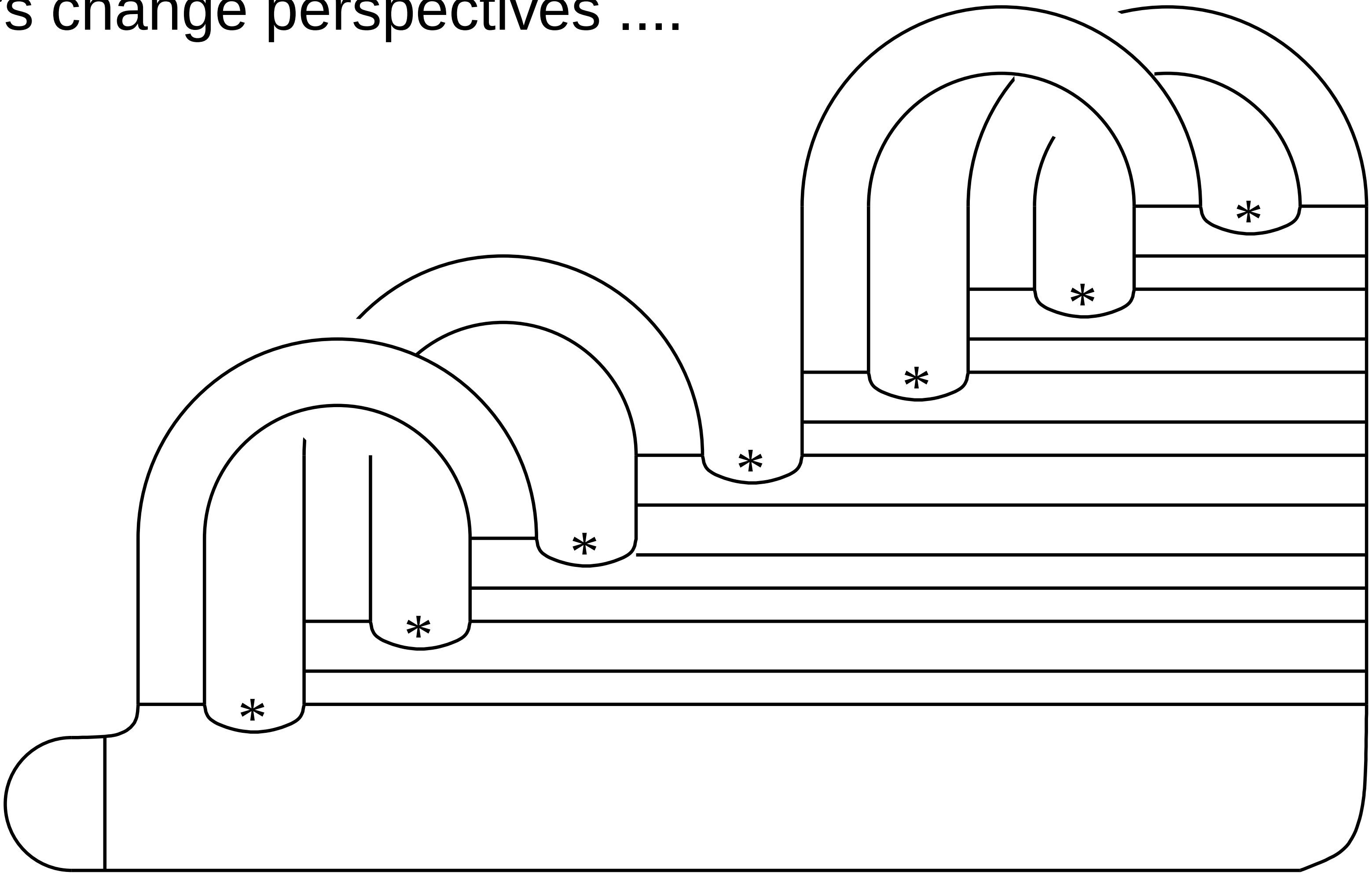
A different perspective

Let's change perspectives ....



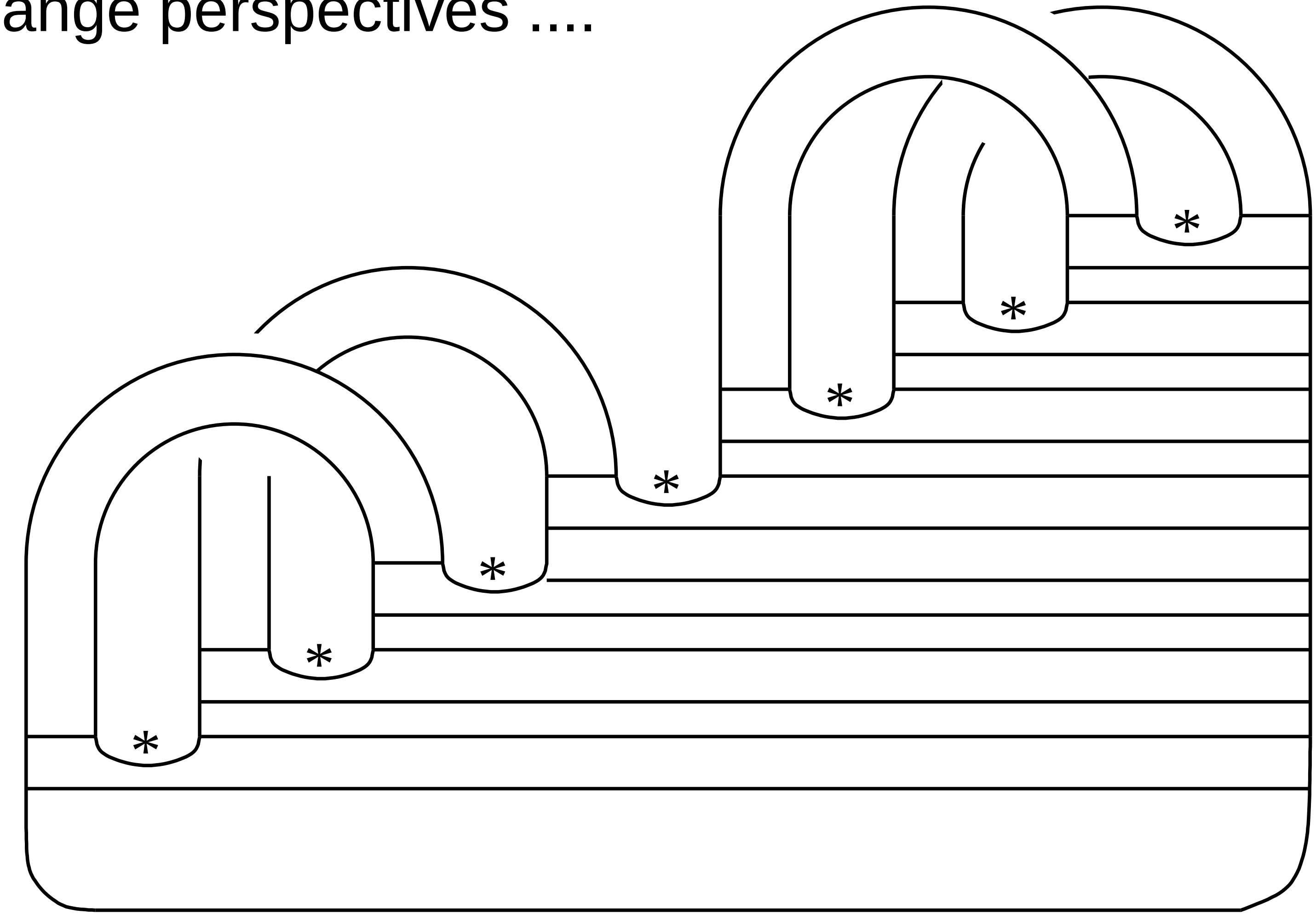
A different perspective

Let's change perspectives ....

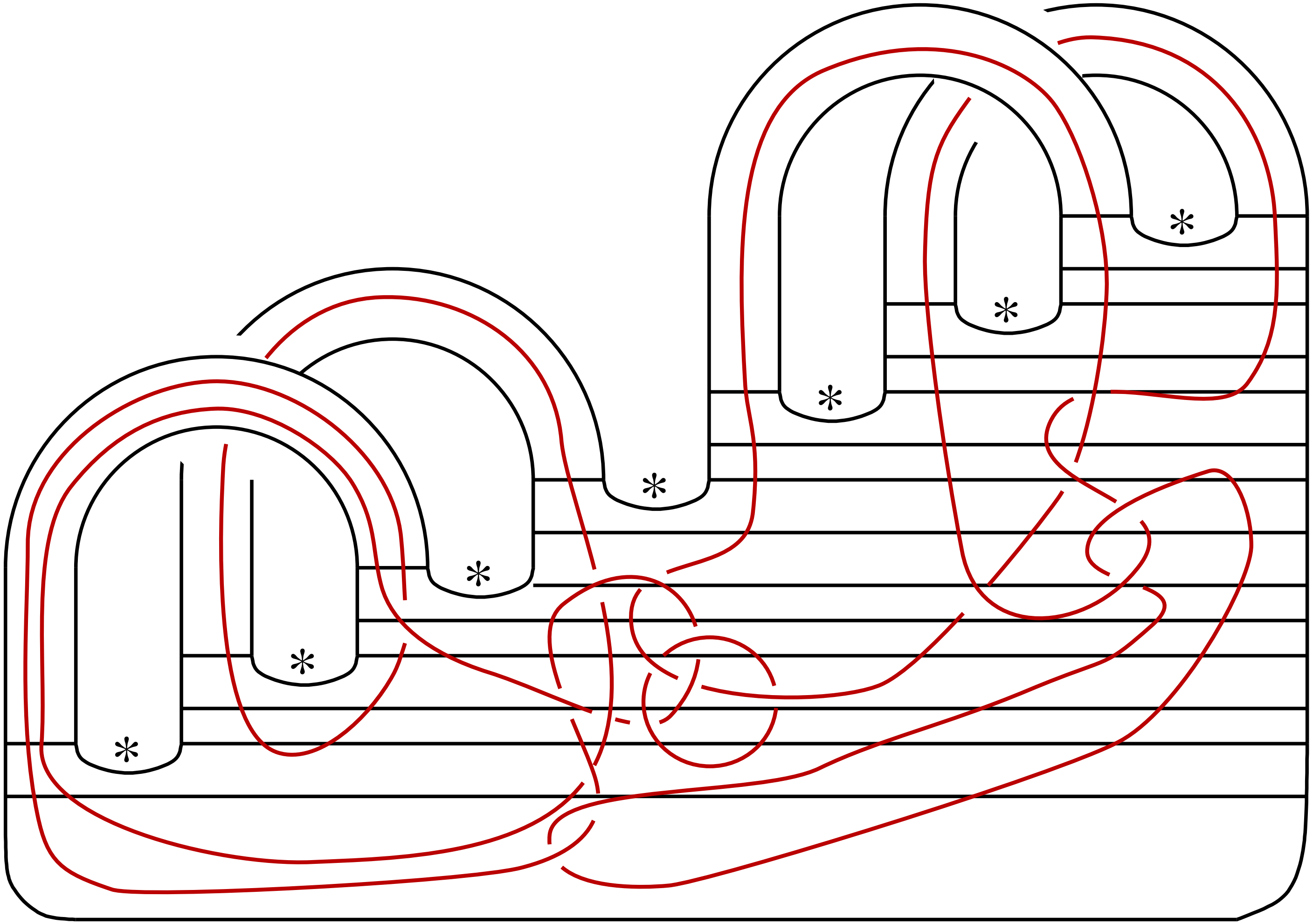


A different perspective

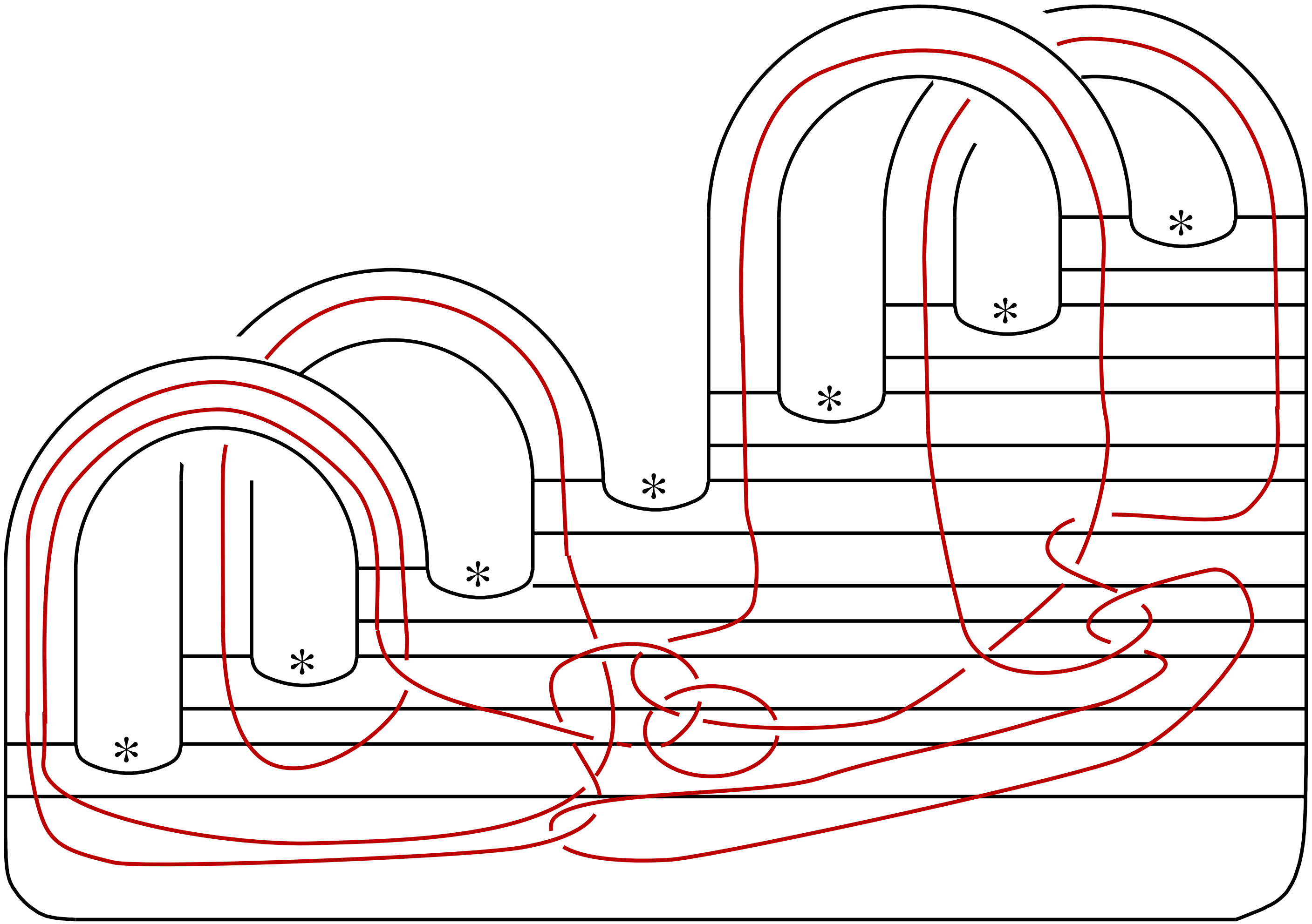
Let’s change perspectives ....



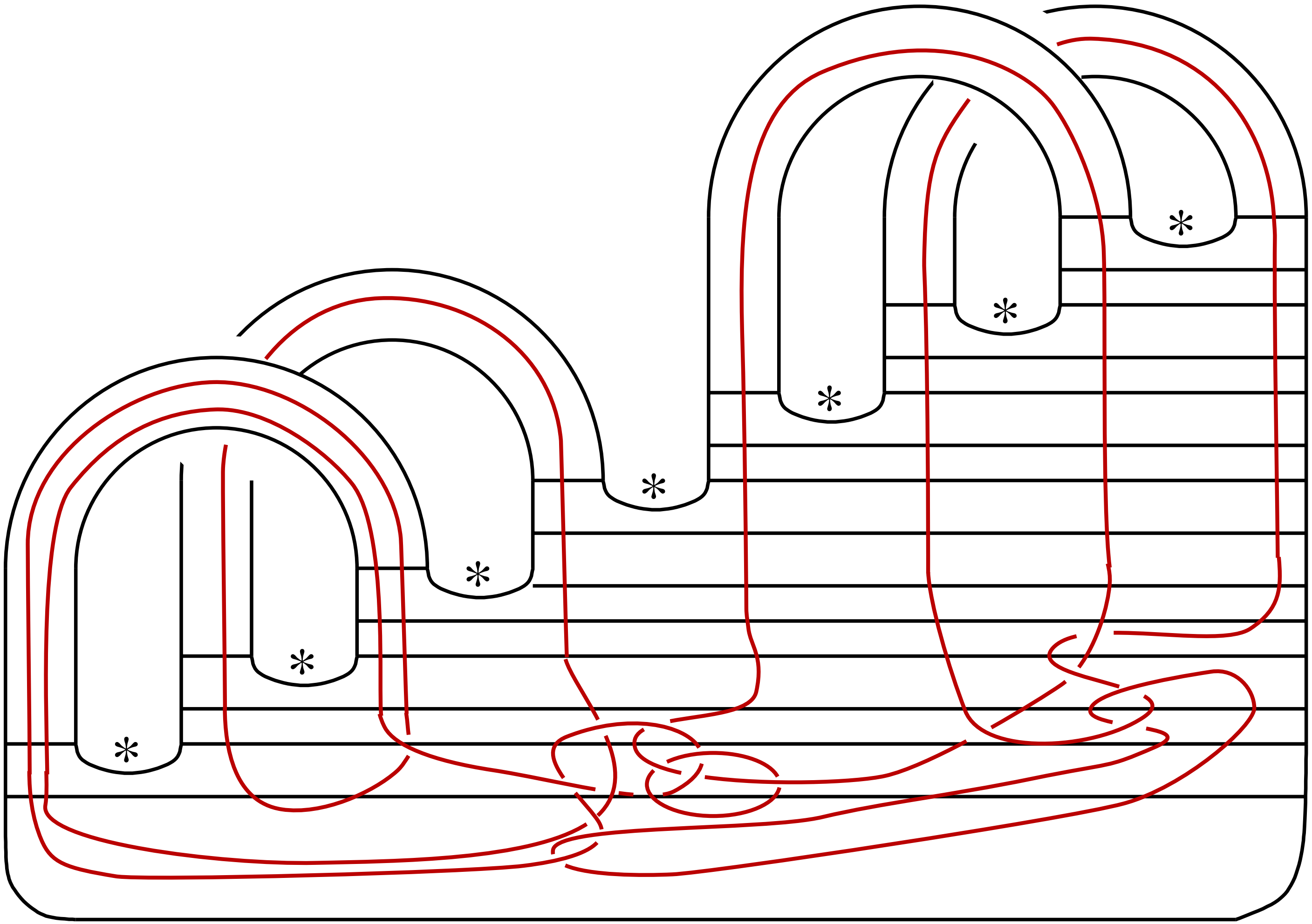
The tangle $\widehat{T}_g$



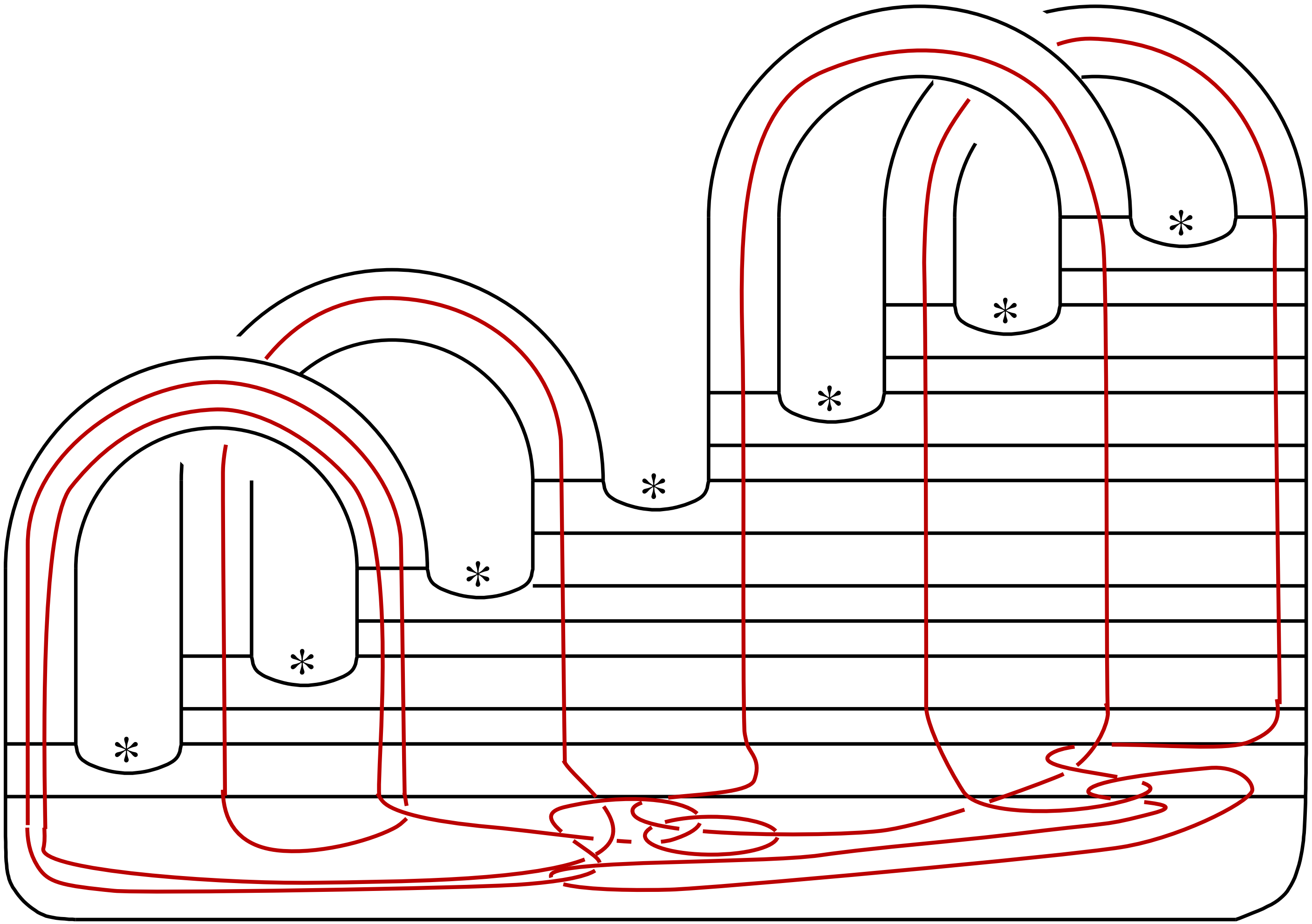
The tangle $\widehat{T}_g$



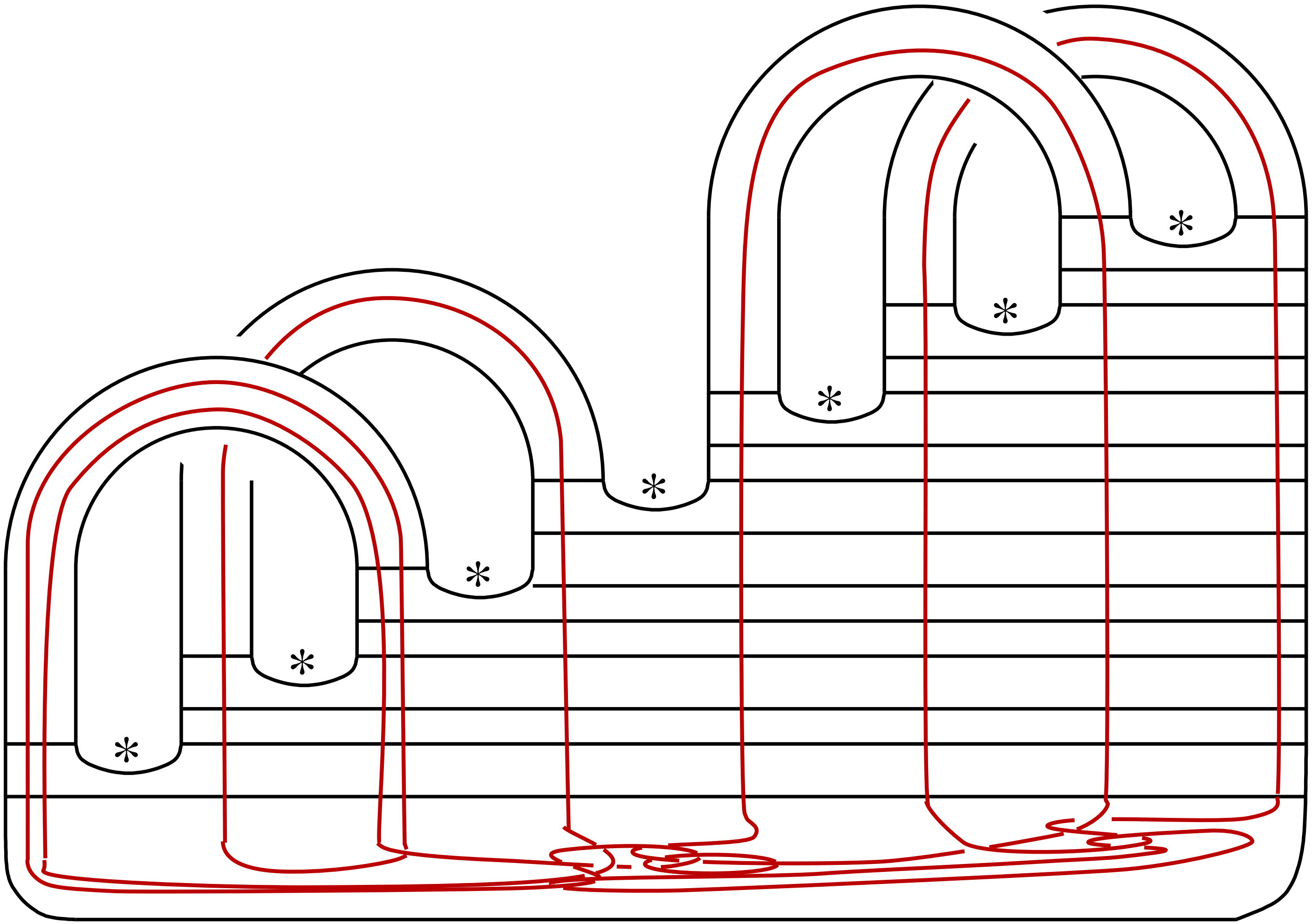
The tangle $\widehat{T}_g$



The tangle $\widehat{T}_g$

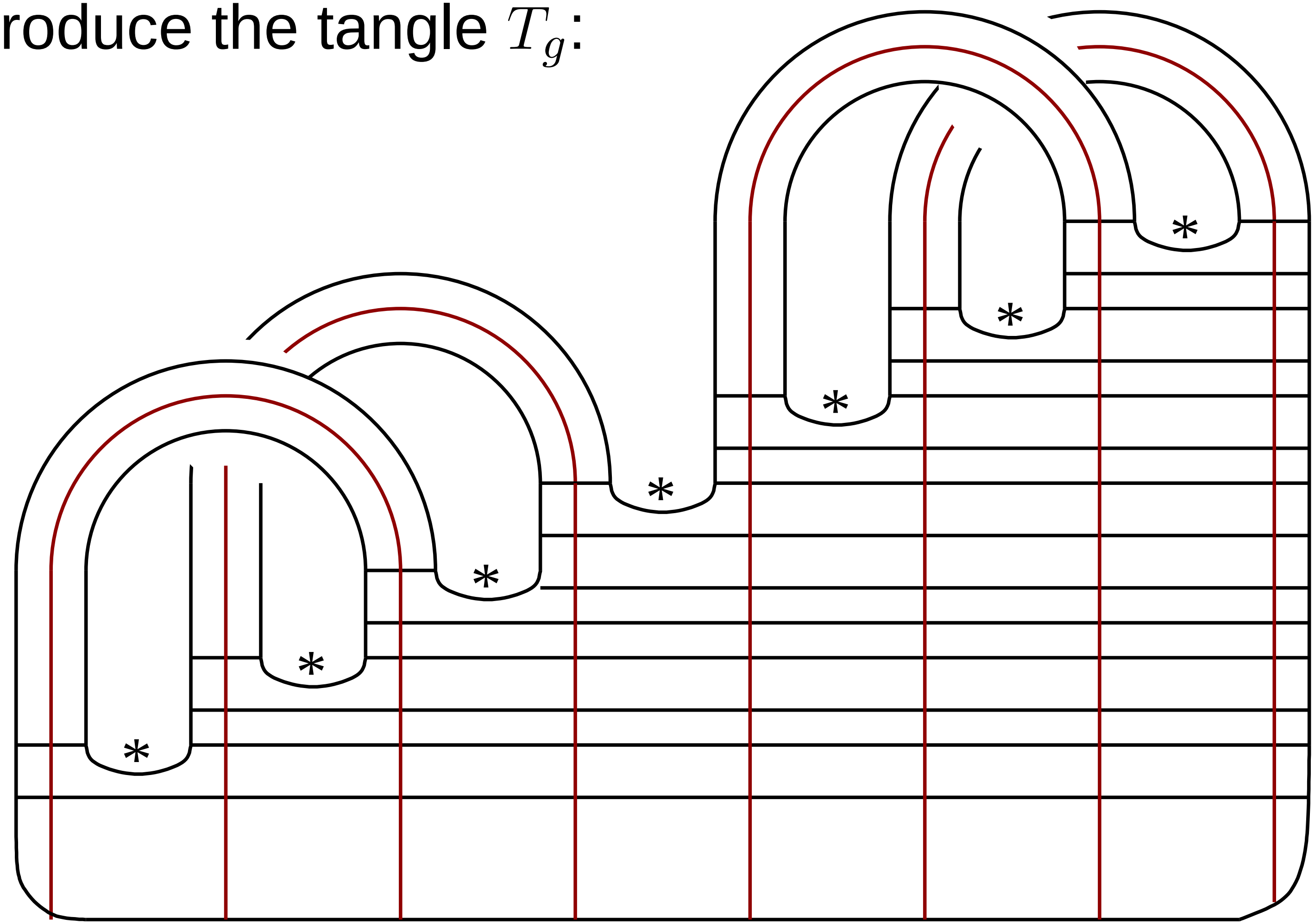


The tangle $\widehat{T}_g$



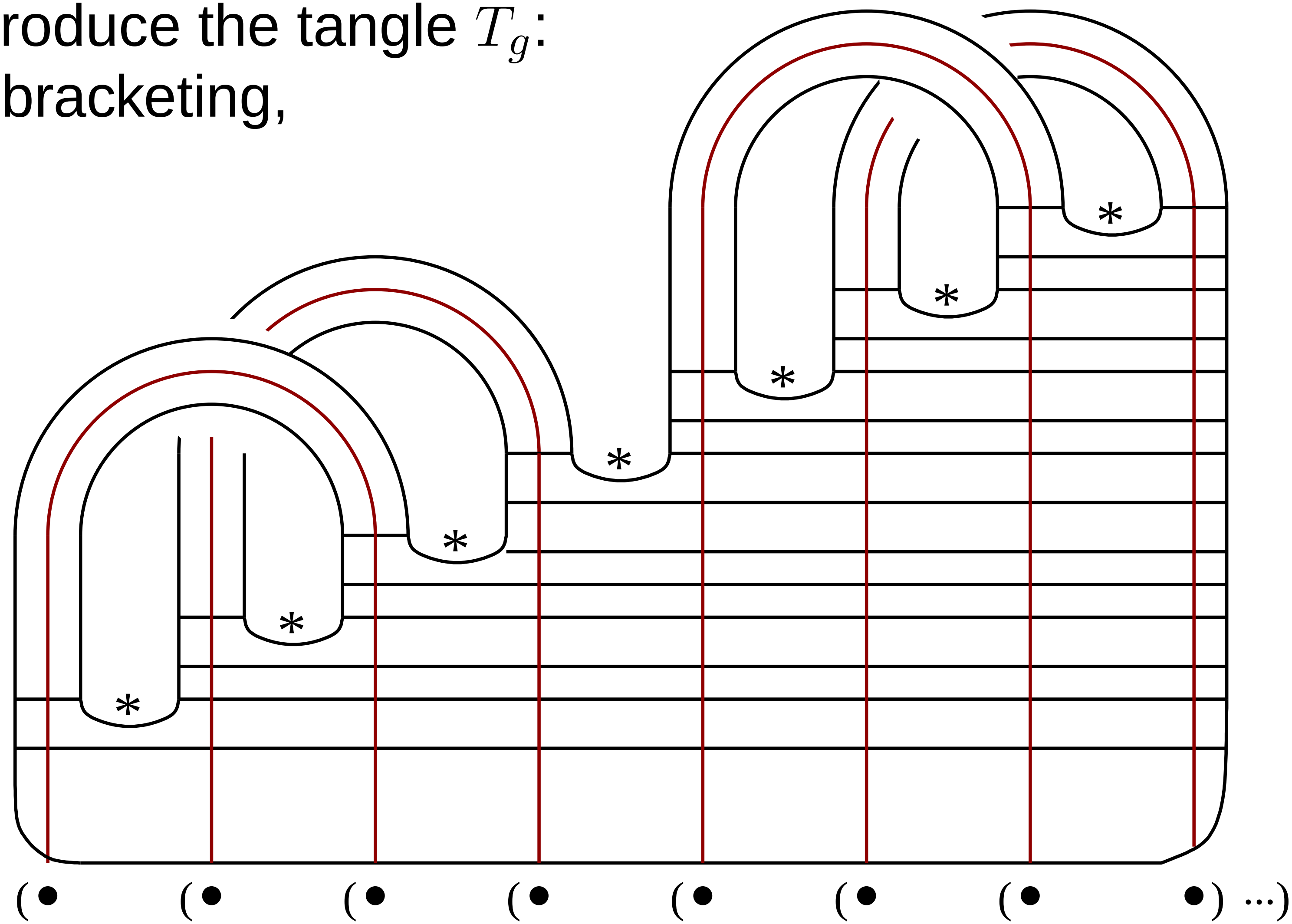
The tangle $\widehat{T}_g$

Now, introduce the tangle $\widehat{T}_g$:



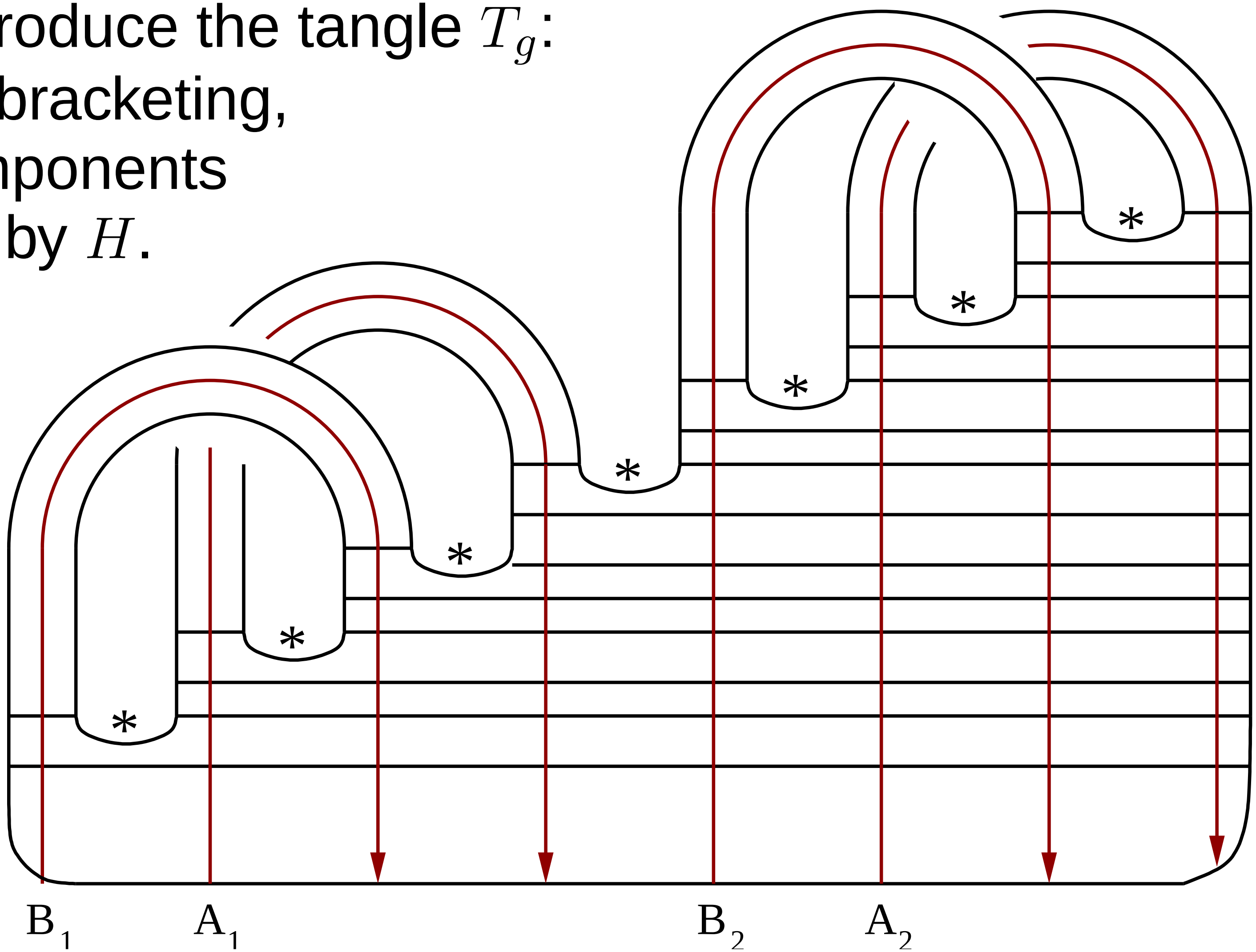
The tangle $\widehat{T}_g$

Now, introduce the tangle $\widehat{T}_g$:
With its bracketing,



The tangle $\widehat{T}_g$

Now, introduce the tangle $\widehat{T}_g$:
With its bracketing,
and components
labelled by H.



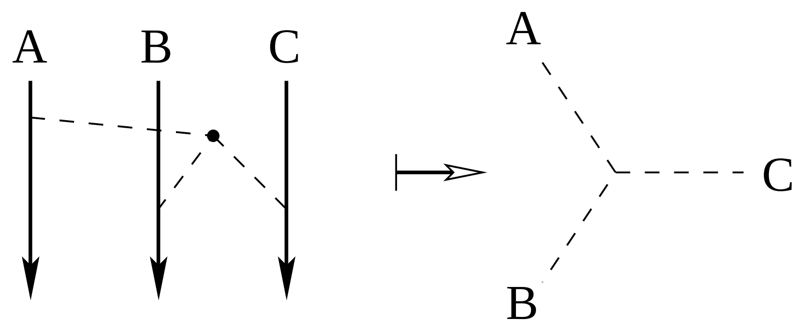
Y-graph part of the AMR invariant of $\widehat{T}_g$.

For any $G \hookrightarrow \Sigma_{g,1}$, we can apply AMR to $\widehat{T}_g$:

$$Z_G(\widehat{T}_g) \in \mathcal{A}(\uparrow^{2g}), \quad Z_{G^{st}}(\widehat{T}_g) = 1.$$

Using the labeling of the components of $\widehat{T}_g$ by a symplectic basis $\{A_i, B_i\}_{i=1}^g$ for H, we can construct a map

$$\mathcal{A}(\uparrow^{2g}) \to D_1(H_{\mathbb{Q}}) \cong \Lambda^3 H_{\mathbb{Q}}$$



Realization Theorem

Composing these, we define a map

$$G \mapsto Z^Y_G(\widehat{T}_g) \in \Lambda^3 H_{\mathbb{Q}}.$$

Theorem (ABMP)**.** *The AMR invariant realizes the first Johnson homomorphism. In particular, for any* $G \hookrightarrow \Sigma_{g,1}$*,*

$$\tau_1(\varphi) = 4\left(Z^Y_{\varphi(G)}(\widehat{T}_g) - Z^Y_G(\widehat{T}_g) \right)$$

for $\varphi \in \mathcal{I}_{g,1}$*.*

Proof. Show that the map $\mathcal{I}_{g,1} \to \Lambda^3 H_{\mathbb{Q}}$ is an $MC_{g,1}$-equivariant homomorphism and compute it on a generator of $\mathcal{I}_{g,1}$. □

Or...

Ptolemy representation of τ_1

Theorem (ABMP)**.** *The AMR invariant realizes (one fourth) the fatgraph extension of the first Johnson homomorphism to the Ptolemy groupoid as introduced by Morita-Penner.*

In particular, for every Whitehead move on a marked bordered fatgraph $W: G \to G'$, we obtain an element

$$Z^Y(\widehat{T}_g)(W) = Z^Y_{G'}(\widehat{T}_g) - Z^Y_G(\widehat{T}_g) \in \Lambda^3 H_{\mathbb{Q}}$$

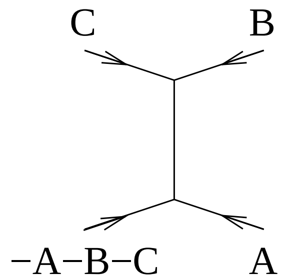
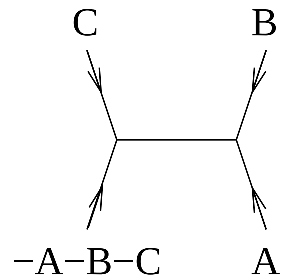
such that for any sequence of moves representing $\varphi \in \mathcal{I}_{g,1}$,

$$G = G_0 \xrightarrow{W_1} G_1 \to \cdots \xrightarrow{W_k} G_k = \varphi(G),$$

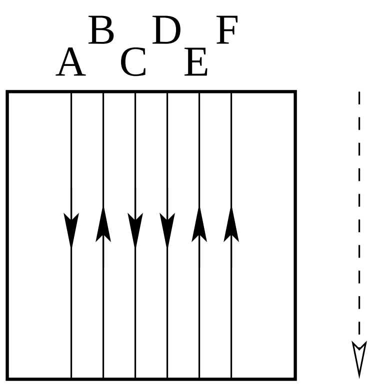
$$\tau_1(\varphi) = 4 \sum_i Z^Y(\widehat{T}_g)(W_i).$$

Proof

We must find the change of $Z^Y(\widehat{T}_g)$ under a Whitehead move:

$Z^Y(\widehat{T}_g)$ () $-$ $Z^Y(\widehat{T}_g)$ () $=?$

where the labels of the oriented edges in the figure represent the signed sums of H-labels of components of $\widehat{T}_g$:

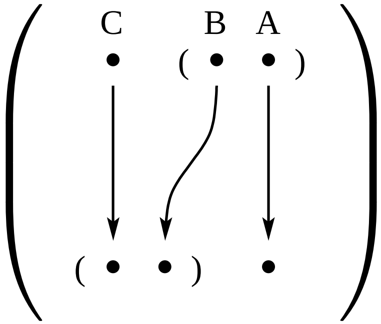


$$\Rightarrow A - B + C + D - E - F$$

Proof

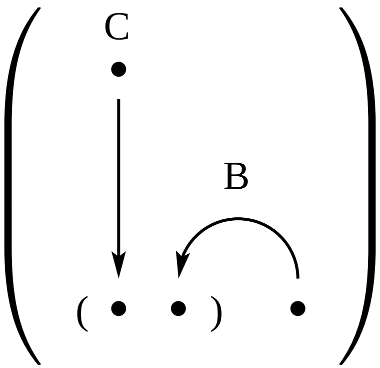
It is easy to check that:

Z^Y

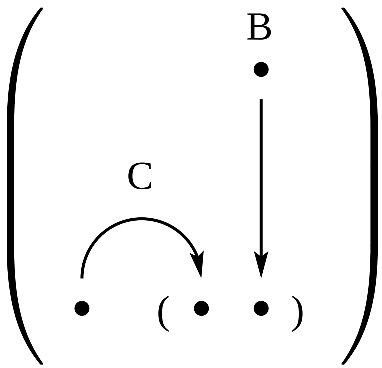


$$= -\frac{1}{24} A \wedge B \wedge C$$

Z^Y



$= Z^Y$

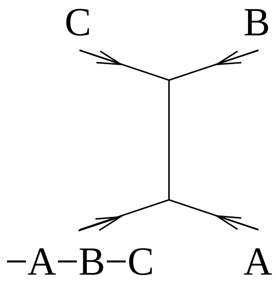


$= 0.$

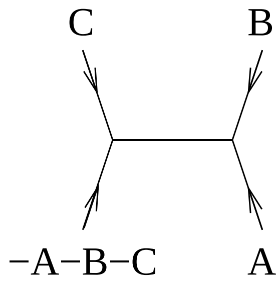
Proof

So we have:

$$Z^Y(\widehat{T}_g)\left(\right.$$



$$\left.\right) - Z^Y(\widehat{T}_g)\left(\right.$$



$$\left.\right)$$

$$= \frac{1}{4}\left(\frac{1}{6} A \wedge B \wedge C\right),$$

which exactly matches (one fourth) the formula of Morita-Penner, thus it is an extension of τ_1. In particular, it is an extension of τ_1 to all of $MC_{g,1}$.

What's next?

There are still some questions:

- Is there a geometric argument for why this gives τ_1?
- What about the higher Johnson homomorphisms τ_k? How would this relate to the fatgraph Ptolemy extensions given by B-Kawazumi-Penner?
- How does this relate to the universal finite type invariants of homology cylinders?

Fin

Thank you.