Braids, self-distributivity and Garside categories

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- The Garside structure of braids is the emerging part of an iceberg: the Garside structure of self-distributivity.

- Revisiting old results of 1985–95 about the self-distributive law in the new context of Garside categories.

Plan :

- The Garside structure of braids
- The normal form in a Garside monoid
- Garside categories
- The category $\mathcal{LD}^+$ of self-distributivity
- The Embedding Conjecture

The Garside structure of braids

every element of B_n can be expressed as $a^{-1}b$ with a, b in B_n^+

↓

Proposition 1 (Garside, 1967):

- The braid group B_n is a group of fractions for the monoid B_n^+.

↑

the monoid defined by the presentation...

Proposition 2 (Garside, 1967):

- The monoid B_n^+ is cancellative: $abc = ab'c$ implies $b = b'$;
- It admits gcd's and lcm's w.r.t. divisibility: $a \preccurlyeq b$ if $\exists c (ac = b)$;
- Every bounded $\prec$-ascending chain is finite;
- The left and right divisors of Δ_n coincide, and generate B_n^+.

↑

half-turn on n strands: $\Delta_1 = 1$, $\Delta_n = \Delta_{n-1}\sigma_{n-1}...\sigma_1$

The greedy normal form

- Main (?) application of Garside's results: a good normal form.
  (Adjan, ElRifai–Morton, Thurston, ...)

- Simple n-braids = divisors of Δ_n ($\leftrightarrow$ permutations of $1, ..., n$)

- **Lemma**: Every braid a in B_n^+ admits one maximal simple divisor.

  $\uparrow$ the head $H(a)$ of a, namely $\gcd(a, \Delta_n)$

- Then

$$a = H(a) \cdot a' = H(a) \cdot H(a') \cdot a'' = H(a) \cdot H(a') \cdot H(a'') \cdot ... = ...$$

$\rightsquigarrow$ a distinguished decomposition of a in terms of simple braids (*i.e.*, of permutations): the greedy normal form.

- Extension from B_n^+ to B_n: express an arbitrary braid
  - either as $a^{-1}b$ with a, b positive and left-coprime,
  - or as $\Delta_n^{-k}b$ with b positive not left-divisible by Δ_n,

  and use the NF of a and b.

The greedy NF is local

- **Why is the greedy NF good? Because it is easy:**
  - to recognize normal sequences,
  - to compute the NF's of $a\sigma_i$ and $\sigma_i a$ from the NF of a.

- **Proposition 1:** A sequence $(a_1, ..., a_d)$ of simple n-braids is normal iff, for each $r < d$, the subsequence (a_r, a_{r+1}) is normal.

- Proof: Assume (a_1, a_2) and (a_2, a_3) normal.

Want: (a_1, a_2, a_3) is normal, *i.e.*, $a_1 = H(a_1a_2a_3)$ or, equivalently,

$$a \text{ simple} \preccurlyeq a_1a_2a_3 \implies a \preccurlyeq a_1.$$

Assume $aa' = \mathrm{lcm}(a, a_1)$. Then, for each x,

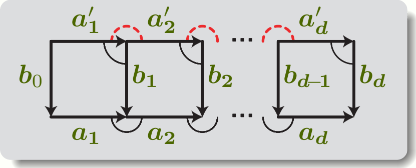
$$a \preccurlyeq a_1x \;\Leftrightarrow\; \mathrm{lcm}(a, a_1x) = a_1x \;\Leftrightarrow\; a' \preccurlyeq x.$$

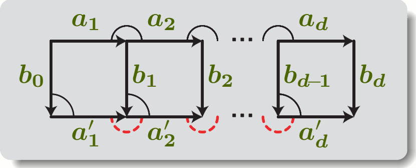
So $a \preccurlyeq a_1a_2a_3 \Leftrightarrow a' \preccurlyeq a_2a_3 \Rightarrow a' \preccurlyeq a_2 \Leftrightarrow a \preccurlyeq a_1a_2 \Rightarrow a \preccurlyeq a_1$. $\square$

$$\begin{tikzcd}
\cdot \arrow[r, "a_1"] \arrow[d, "a"'] \arrow[dr, phantom, "\vee"] & \cdot \arrow[r, "a_2"] \arrow[d, "1"] \arrow[dr, phantom, "\vee"] & \cdot \arrow[r, "a_3"] \arrow[d, "1"] \arrow[dr, phantom, "\vee"] & \cdot \arrow[d, "1"] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Multiplying a NF

- **Proposition 2:** Assume $\mathrm{NF}(a) = (a_1, \ldots, a_d)$ and b is simple. Then
  - $\mathrm{NF}(ba) = (a'_1, \ldots, a'_d, b_d)$,
    where $b_0 = b$ and $(a'_r, b_r) = \mathrm{NF}(b_{r-1} a_r)$ for $r \geqslant 1$,
  - $\mathrm{NF}(ab) = (b_0, a'_1, \ldots, a'_d)$,
    where $b_d = b$ and $(b_{r-1}, a'_r) = \mathrm{NF}(a_r b_r)$ for $r \geqslant 1$.



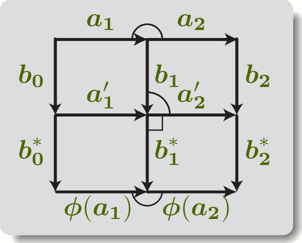


Proof for multiplication on the left

$$\begin{tikzcd}
{} \arrow[r, "a_1'"] \arrow[d, "b_0"'] & {} \arrow[r, "a_2'"] \arrow[d, "b_1"] & {} \arrow[d, "b_2"] \\
{} \arrow[r, "a_1"'] & {} \arrow[r, "a_2"'] & {}
\end{tikzcd}$$

- **Want:** (a_1', a_2') is normal.
  Assume $a \preccurlyeq a_1' a_2'$.
    Then $a \preccurlyeq a_1' a_2' b_2 = b_0 a_1 a_2$.
      Assume $aa' = \operatorname{lcm}(a, b_0)$.
        Then (as before) $a' \preccurlyeq a_1 a_2$.
          As (a_1, a_2) is normal, $a' \preccurlyeq a_1$.
            Hence $a \preccurlyeq b_0 a_1 = a_1' b_1$.
              As (a_1', b_1) is normal, $a \preccurlyeq a_1'$. □

Proof for multiplying on the right



- **Want:** (a'_1, a'_2) is normal.

For x simple, introduce x^* s.t. $xx^* = \mathbf{\Delta}$, and let $\phi(x) = x^{**}$.
Then $x\mathbf{\Delta} = \mathbf{\Delta}\phi(x)$, and ϕ is an automorphism.
Hence $(\phi(a_1), \phi(a_2))$ is normal.
Next (b_1, a'_2) normal $\Rightarrow \mathrm{gcd}(b^*_1, a'_2) = 1$ (actually $\Leftrightarrow$).
Assume $a \preccurlyeq a'_1 a'_2$.
Then $a \preccurlyeq a'_1 a'_2 b^*_2 = b^*_0 \phi(a_1)\phi(a_2)$.
Hence (same argument as before) $a \preccurlyeq b^*_0 \phi(a_1) = a'_1 b^*_1$.
Hence $a \preccurlyeq \mathrm{gcd}(a'_1 b^*_1, a'_1 a'_2) = a'_1$. □

- Does not work only for B_n, B_n^+, and Δ_n (D.–Paris '97, ...)

- Definition: (M, Δ) is a weakly left-Garside monoid if
  - M is a left cancellative monoid,
  - M admits right lcm's and left gcd's,
  - every bounded $\prec$-ascending sequence is finite,
  - Δ belonds to M, the left divisors of Δ generate M and include the right divisors of Δ (equivalently: are closed under $\backslash$).

  (M, Δ) is strongly left-Garside if, in addition,
  - M is right cancellative (equivalently: ϕ is injective).

$\uparrow$

where $aa^* = \Delta$ and $\phi(a) = a^{**}$

- Then the construction of the greedy NF extends:
  - Prop. 1 and 2 hold for every weakly left-Garside monoid;
  - Prop. 3 holds for every strongly left-Garside monoid.

$\rightsquigarrow$ bi-automatic structure when finitely many simples

- **Theorem** (**Garside**, 1967) (B_n^+, Δ_n) is a Garside monoid.

↑ left-Garside and right-Garside with the same Δ

- Many other examples:
  - Spherical Artin–Tits groups (**Charney**),
  - Dual monoids (**Birman–Ko–Lee**, **Bessis**, ...),
  - Certain complex reflection groups (**Bessis**, **Corran**, ...),
  - Free groups (**Bessis**, **Brady–Crisp–Kaul–McCammond**, ...),
  - Many more... (**Picantin**, **Krammer**, ...).

- Many questions:
  - Conjugacy problem (**Gonzalez-Meneses**, **Gebhardt**, **Lee–Lee**),
  - Properties of roots (**Sibert**, **Gonzalez-Meneses**, **Lee–Lee**),
  - Other normal forms (**Burckel**, **Fromentin**, ...).

Extension 2

- Replace monoids with categories:

  (Krammer, Digne–Michel, Bessis, 2005-6)

  implicit in (Deligne, 1971), maybe in (D., ∼1990).

- Principle: Keep the diagrams, but add objects:
  - call the elements of the monoid morphisms,
  - attach a source $\partial_0 f$ and a target $\partial_1 f$ with every morphism f.

Before: $a \rightsquigarrow \xrightarrow{a}$

After: $a \rightsquigarrow x \xrightarrow{a} y$

source target

- Benefit:
  - a partial product: fg exists only if $\partial_1 f = \partial_0 g$,
  - a local notion of simple: for each object x, the simples at x.

- Of course: monoid = category with one object.

Left-Garside categories

- Definition: $(\mathcal{C}, \Delta)$ is a **weakly left-Garside** category if
  - $\mathcal{C}$ is a category and $\mathcal{H}om(\mathcal{C})$ is left-cancellative,
  - any two morphisms with the same source have a right lcm,
  - every $\prec$-ascending sequence in $\mathrm{Div}(f)$ (left-divisors of f) is finite,
  - Δ maps $\mathcal{O}bj\,(\mathcal{C})$ to $\mathcal{H}om(\mathcal{C})$ so that $\Delta(x) \in \mathrm{Hom}(x, -)$,

every nontrivial element of $\mathrm{Hom}(x, -)$ is left-divisible by a nontrivial left-divisor of $\Delta(x)$, and every right-divisor of $\Delta(x)$ is simple.

↑

simple at x

- If $(\mathcal{C}, \Delta)$ is weakly left-Garside, there is a (unique) functor ϕ s.t.

$$\phi(x) = \partial_1 \Delta(x), \qquad f\Delta(y) = \Delta(x)\phi(f) \text{ for } f : x \to y.$$

(so Δ is a natural transformation of id to ϕ)

- Definition: $(\mathcal{C}, \Delta)$ is **strongly left-G.** if, moreover, ϕ is injective.

- Equivalently: ϕ is injective on objects, and $\mathcal{H}om(\mathcal{C})$ is right-cancellative.

Garside categories

- Garside = strongly left-Garside + right-Garside with the same Δ.
- Examples:
  - For M a (left)-Garside monoid:

$$\mathcal{O}bj\,(\mathcal{C}_M) = \{1\},\ \ \mathcal{H}om(\mathcal{C}_M) = \{1\}\times M\times\{1\},\ \ \Delta(1) = \Delta.$$

or

$$\mathcal{O}bj\,(\widetilde{\mathcal{C}}_M) = M,\ \ \mathcal{H}om(\widetilde{\mathcal{C}}_M) = \{(a,b,c) \mid ab = c\},\ \ \Delta(1) = \Delta.$$

  - (Krammer) MCG's of disks with punctures on the boundary,
  - (Godelle) Ribbon categories,
  - (Digne–Michel) Conjugacy categories,
  - (Bessis) Divided categories,
  - Braid category $\mathcal{B}^+$

$$\mathcal{O}bj\,(\mathcal{B}^+) = \mathbb{Z}_+,\ \mathcal{H}om(\mathcal{B}^+)=\{(n,a,n) \mid a\in B_n^+\},\ \Delta(n) = \Delta_n.$$

or

$$\mathcal{O}bj\,(\widetilde{\mathcal{B}}^+)=\mathrm{Seq}(\mathbb{N}),\ \mathcal{H}om(\widetilde{\mathcal{B}}^+)=\{(s,a,s\bullet a) \mid a\in B_n^+\},\ \Delta(s)=\Delta_{|s|}.$$

  - Self-distributivity.

- The left self-distributive law LD:

$$x(yz) = (xy)(xz).$$

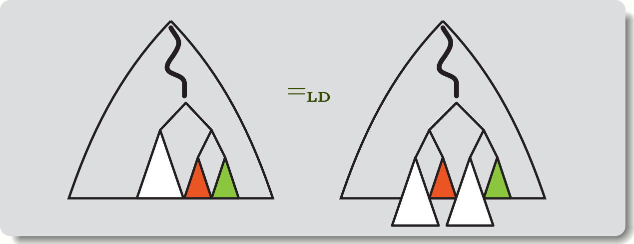
- Examples of LD-systems:
  - $x * y = f(y)$, in any set,
  - $x * y = xyx^{-1}$, in a group,
  - $x * y = (1-t)x + ty$, in a $\mathbb{Z}[t]$-module,
  - $x * y = x$ applied to y, elementary embeddings (cannot be proved to exist: large cardinal required),
  - $x * y = x\,\mathrm{sh}(y)\,\sigma_1\,\mathrm{sh}(x)^{-1}$, in B_∞ with $\mathrm{sh}(\sigma_i) = \sigma_{i+1}$,
  - Laver tables (an inverse system consisting of LD-systems with $1, 2, 4, 8, \ldots$ elements),
  - Free LD-systems.

- **Fact** (trivial): The free LD-system of rank n is $T_n/{=_{\mathrm{LD}}}$,

where

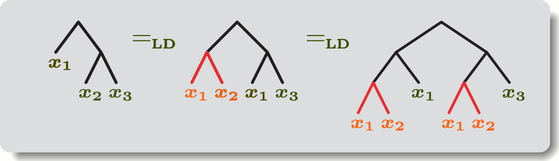
T_n is the set of all **terms** (=bracketed expressions) on $x_1, ..., x_n$,
equipped with the operation $t_1 * t_2 = (t_1)(t_2)$,
"absolutely free algebra—or free magma—of rank n"
—also seen as **rooted binary trees** with labeled leaves

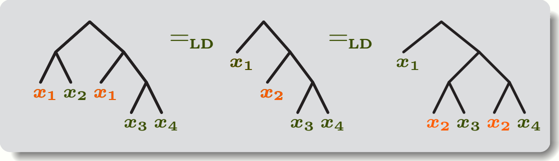
$=_{\mathrm{LD}}$ is the smallest congruence on T_n that contains all pairs
$$(t_1 * (t_2 * t_3)\,,\,(t_1 * t_2) * (t_1 * t_3)).$$



LD-equivalence

- More examples of LD-equivalences:



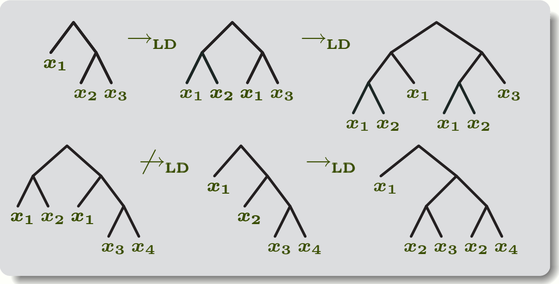


... and everything obtained by substituting variables with terms.

LD-expansion

- How to study the relation $=_{\mathrm{LD}}$ (a very complicated object) ?
  Orientate it. $\simeq$ Garside: study B_n by considering B_n^+

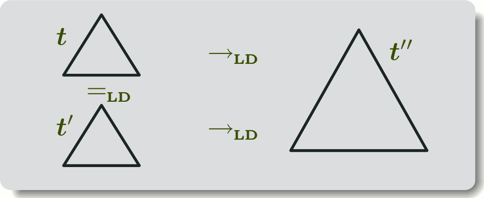
- **Definition:** t' is an **LD-expansion** of t, denoted $t \rightarrow_{\mathrm{LD}} t'$, if t' can be obtained from t by applying LD in the expanding direction only.



- Clearly, $=_{\mathrm{LD}}$ is generated by $\rightarrow_{\mathrm{LD}}$:
  $t =_{\mathrm{LD}} t'$ holds iff there exists a $\rightarrow_{\mathrm{LD}}$-zigzag from t to t'.

The confluence theorem

Theorem (D., ’86) Two terms are LD-equivalent
iff they admit a common LD-expansion.



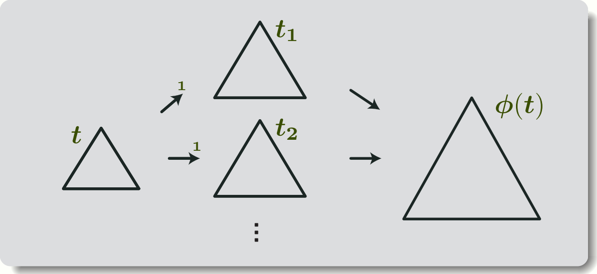
Corollary (’91) Braid groups are orderable.

- Proof of Corollary: - The free LD-system of rank 1 is orderable, because any two terms of T_1 are comparable w.r.t. the relation "being LD-equivalent to an iterated left subterm of";
  - Then use the latter LD-system to color the strands of braids. ☐

Proof of the confluence theorem

- How to prove the Confluence Theorem?

- **Main Lemma**: For each t, there exists an LD-expansion $\phi(t)$ of t that is a common LD-expansion of every atomic LD-expansion of t; moreover $t \rightarrow_{\mathrm{LD}} t'$ implies $\phi(t) \rightarrow_{\mathrm{LD}} \phi(t')$.



- From there: $\phi^d(t)$ is a common LD-expansion of all degree d LD-expansions of t. ☐

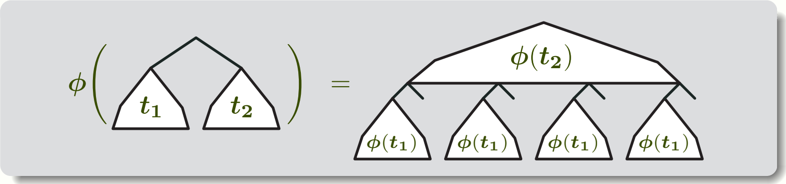
The fundamental expansion $\phi(t)$

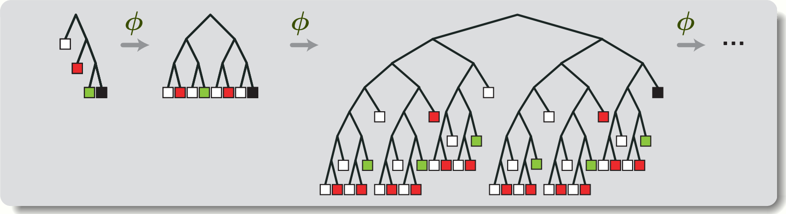
- **Inductive construction of $\phi(t)$:**

$$\phi(x_i) = x_i, \qquad \phi(t_1 * t_2) = \phi(t_1) \circledast \phi(t_2),$$

where $\circledast$ means "distribute once everywhere":

$$t \circledast x = t * x, \qquad t \circledast (t_1 * t_2) = (t \circledast t_1) * (t \circledast t_2).$$





A first attempt

- Is there a left-Garside category here?

- Natural candidate: graph of the LD-expansion relation:

- **Definition**: The category $\mathcal{LD}_0^+$
  $\mathcal{O}bj\,(\mathcal{LD}_0^+) := \{$ terms $\}$,
  $\mathcal{H}om(\mathcal{LD}^+) := \{(t, t') \mid t \to_{\mathrm{LD}} t'\}$.

- Simples at t = all LD-expansions of t between t and $\phi(t)$.

- Least common multiple = least common LD-expansion (??),
  $\rightsquigarrow$ Problem: Not proved to exist...

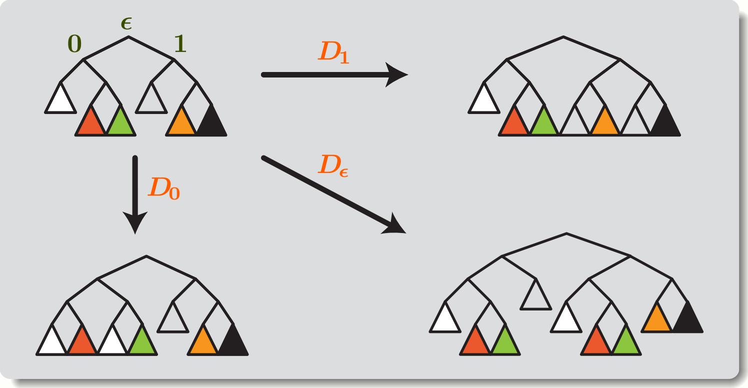
- Solution: Control LD-expansions better:
  $\rightsquigarrow$ Take into account the position where LD is applied.

Naming LD-expansions

- **Attach a label to each atomic LD-expansion:**

  an **address** specifying where LD is applied

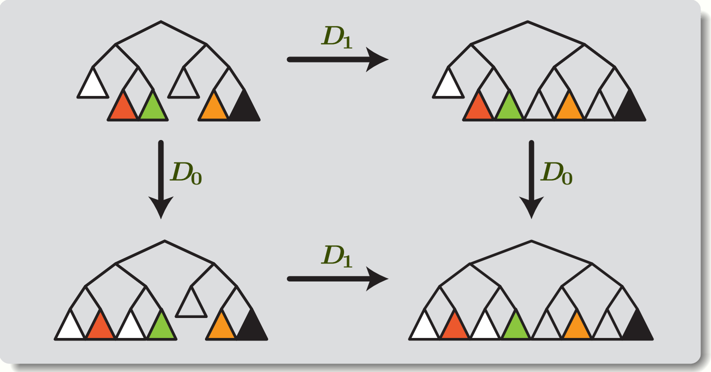
  ⇝ a sequence of 0's and 1's describing the path from the root



LD-relations (1): parallel case

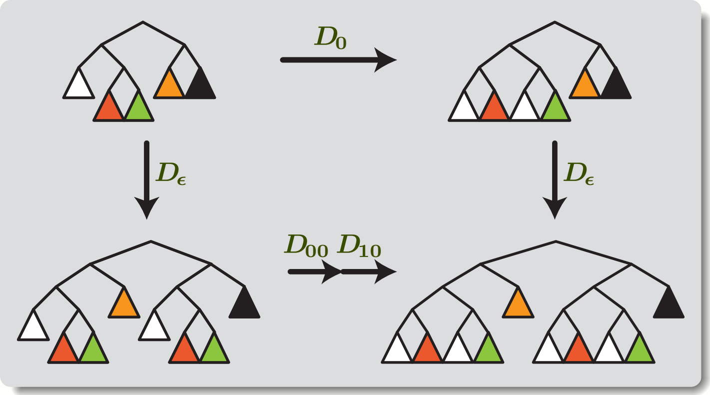
- There are natural relations between the various D_α-expansions.

↑ those coming from (alledgedly) least common LD-expansions



- More generally: $D_\alpha D_\beta = D_\beta D_\alpha$ when α, β are parallel.

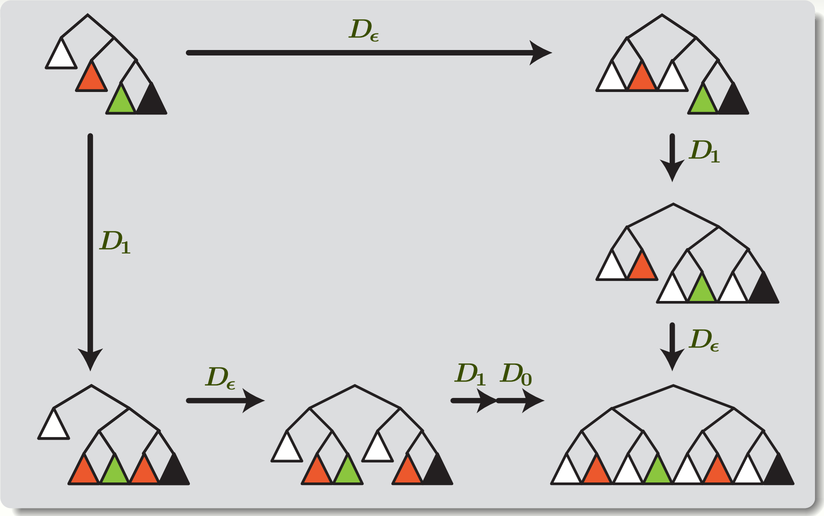
LD-relations (2): nested case



- **More generally:**

$$D_{\alpha 0\beta}\, D_\alpha = D_\alpha\, D_{\alpha 00\beta}\, D_{\alpha 10\beta},$$
$$D_{\alpha 10\beta}\, D_\alpha = D_\alpha\, D_{\alpha 01\beta},$$
$$D_{\alpha 11\beta}\, D_\alpha = D_\alpha\, D_{\alpha 11\beta}.$$

LD-relations (3): critical case



- **More generally:** $D_\alpha\, D_{\alpha 1}\, D_\alpha = D_{\alpha 1}\, D_\alpha\, D_{\alpha 1}\, D_{\alpha 0}$.

- **Definition:** The monoid LD^+:

  $\langle\ \{D_\alpha \mid \alpha$ an address $\}\mid$ LD-relations $\rangle^+$.

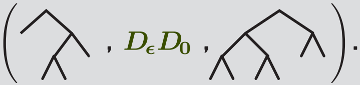
- By construction: a (partial) action of LD^+ on terms via LD-expansions.

- **Definition:** The category $\mathcal{LD}^+$:

  $\mathcal{O}bj\,(\mathcal{LD}^+) := \{$ terms $\}$,

  $\mathcal{H}om(\mathcal{LD}^+) := \{(t, a, t') \mid a \in \mathrm{LD}^+$ and $t \bullet a = t'\}$.

- A typical morphism in $\mathcal{LD}^+$:



The main result

Theorem: The category $\mathcal{LD}^+$ is weakly left-Garside, and there is a projection of $\mathcal{LD}^+$ onto $\mathcal{B}^+$ that preserves the Garside structures.

↑

a surjective lcm-functor

Conjecture: The category $\mathcal{LD}^+$ is strongly left-Garside.

- What we shall do:
  - 1. Explain the connection with braids.
  - 2. Explain why $\mathcal{LD}^+$ is left cancellative and admits lcm's (plus the chain condition);
  - 3. Describe the Δ;
  - 4. Discuss the conjecture.

Projection to braids

- Put $\pi(t)$:= length of the rightmost branch in t,

$$\pi(D_\alpha) := \begin{cases} \sigma_i & \text{for } \alpha = 11\ldots1,\ i-1 \text{ times } 1, \\ 1 & \text{if } \alpha \text{ contains at least one } 0. \end{cases}$$

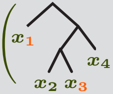
$\pi\Bigg($  $,\ D_\epsilon D_0\ ,$  $\Bigg) = (\ 2\ ,\ \sigma_1\ ,\ 2\ ).$

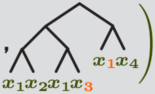
- **Proposition 1:** π is an lcm-functor of $\mathcal{LD}^+$ onto $\mathcal{B}^+$.

- Proof: π(LD-relations) $\subseteq$ braid relations:

$$\pi(D_\alpha\, D_{\alpha 11\beta} = D_{\alpha 11\beta}\, D_\alpha) = (\sigma_i\, \sigma_{i+2+j} = \sigma_{i+2+j}\, \sigma_i),$$
$$\pi(D_\alpha\, D_{\alpha 1}\, D_\alpha = D_{\alpha 1}\, D_\alpha\, D_{\alpha 1}\, D_{\alpha 0}) = (\sigma_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \sigma_{i+1}). \ \square$$

- Also: $\widetilde{\pi} : \widetilde{\mathcal{LD}}^{+} \twoheadrightarrow \widetilde{\mathcal{B}}^{+}$ with $\widetilde{\pi}(t)$ = names of subright variables.

$\widetilde{\pi}\Bigg($ 

$,\ D_\epsilon D_0\ ,$ 

$\Bigg) = ((1,3),\ \sigma_1\ ,(3,1)).$

- **Proposition 2:** $\mathcal{LD}^+$ is left cancellative, and morphisms of $\mathrm{Hom}(t, -)$ with a common right multiple admit a right lcm.

(essentially, properties of the monoid LD^+)

- Proof: The presentation of LD^+ is such that:

  For each pair of generators D_α, D_β, there exists

  one relation of the form $D_\alpha ... = D_\beta ...$ in the presentation.

For such presentations, there exists an effective criterion to decide whether the monoid is left cancellative and admits a right lcm when a common multiple exists: "completeness of right reversing"

Here—as well as for B_n^+—the criterion works. ☐

- For $\prec$-ascending sequences:

  Every chain from t to t' has length $\leqslant \mathrm{size}(t') - \mathrm{size}(t)$.

Delta

- How to define Δ, *i.e.*, how to define simples at t ?

  $\rightsquigarrow$ Use $\phi(t)$: the "fundamental expansion of t",
  expanding all atomic expansions of t

- Define Δ_t in LD^+ as a distinguished way to expand t into $\phi(t)$ following the inductive construction of ϕ:

$$\Delta_t = \begin{cases} 1 & \text{for } t = x, \text{ (size 1 term)} \\ \ldots\Delta_{t_1}\ldots\Delta_{t_2}\ldots & \text{for } t = t_1 * t_2. \end{cases}$$

and then put

$$\Delta(t) = (t, \Delta_t, \phi(t)).$$

- Example:

$$\xrightarrow{\pi} (\quad 3 \quad, \sigma_1\sigma_2\sigma_1, \quad 3 \quad)$$

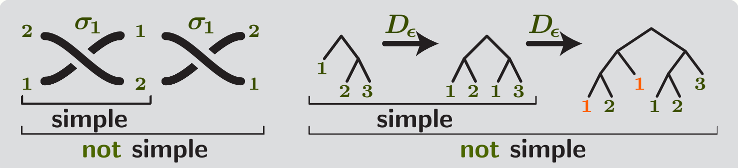
$$\xrightarrow{\widetilde{\pi}} ((1,2,3), \sigma_1\sigma_2\sigma_1, (3,2,1))$$

- **Proposition 3:** (i) If $t \bullet D_\alpha$ exists, then D_α left-divides Δ_t.
  (ii) Right-divisors of Δ_t are simple,
  (iii) Common right-multiples always exist in $\mathrm{Hom}(t, -)$.

- Proof: LD-Relations imply (i), and (ii) implies (iii).

For (ii), remember: a braid a is simple
  iff $\exists!$ an expression $a = \prod_i^< \sigma_{i,e_i}$, with $\sigma_{i,e} = \sigma_i\sigma_{i+1}...\sigma_{i+e-1}$,
  iff $\exists$ a diagram of a in which any two strands cross at most once.

Here: an LD-expansion a is simple
  iff $\exists!$ an expression $a = \prod_\alpha^< D_{\alpha,e_\alpha}$, with $D_{\alpha,e} = D_\alpha D_{\alpha 1}...D_{\alpha 1^{e-1}}$,
  iff $\exists$ a term in the target of a in which no x_i **covers** itself. ☐



- **Conjecture**: The category $\mathcal{LD}^+$ is strongly left-Garside.

i.e., the functor ϕ is injective—“fully left-Garside”

- ϕ is injective on terms; the problem is for morphisms.

- Equivalent forms:

  - The monoid LD^+ admits right cancellation.
  - Any LD-equivalent terms admit a least common LD-expansion.
  - For all terms t, t', the cardinality of $\mathrm{Hom}_{\mathcal{LD}^+}(t, t')$ is at most 1.
  - The category $\mathcal{LD}^+$ is isomorphic to $\mathcal{LD}_0^+$ (graph of LD-expansion).
  - The functor ϕ preserves normality.

The Embedding Conjecture (cont'd)

- **Partial instances proved** (*e.g.*, ϕ injective on simple morphisms).

- Possible attack:

- Enough to show: ϕ preserves left-coprimeness of simples.
  - A fortiori, enough to show for a, b simple:

$$\boldsymbol{\phi(\mathrm{gcd}(a, b)) = \mathrm{gcd}(\phi(a), \phi(b)).}$$

  - Simples admit **unique** expressions ("permutation-expansions"):

$$a = \prod_{\alpha \text{ address}}^{<} \underbrace{D_{\alpha} D_{\alpha 1} D_{\alpha 11} \cdots}_{e(\alpha) \text{ factors}}.$$

  - Use the sequence of $e(\alpha)$'s as coordinates for a and find explicit formulas for the coordinates of $\phi(a)$ and $\mathrm{gcd}(a, b)$.

—not so easy, even for braids...

- **Originally (1991): Solve the word problem of LD.**

  (= find an algorithm that recognizes whether $t =_{\mathrm{LD}} t'$ is true)

  an unprovable set-theoretical assumption

  $\downarrow$

Theorem (Laver, 1989): If there exists a self-similar rank, then the word problem of LD is decidable.

The point: one needs an orderable LD-system.
Set Theory provides a hypothetic orderable LD-system.
The Garside structure of $\mathcal{LD}^+$ shows free LD-systems are orderable.
(+ braid applications)

- **Now: shorter proofs of braid acyclicity (Larue, Dynnikov) provide shorter proofs for the orderability of free LD-systems.**

- **What remains? The Garside structure of $\mathcal{LD}^+$ as an explanation for the Garside structure of braids ("π is an lcm-functor").**

- **At least: an example of a left-Garside, non-Garside structure.**

Associativity

- Everything is similar when associativity replaces selfdistributivity:

$$\boxed{x(yz) = (xy)z}$$

  - A-expansion: replace $t_1 * (t_2 * t_3)$ with $(t_1 * t_2) * t_3$;
  - monoid A^+: generated by all A_α with A-relations (MacLane);
  - category $\mathcal{A}^+$: $\mathcal{H}om(\mathcal{A}^+) = \{(t, g, t') \mid g \in \mathrm{A}^+ \text{ and } t \bullet g = t'\}$.

- **Good news: There is a Garside structure.**
  - Least common A-expansions exist: the Tamari lattice;
  - The monoid A^+ is Garside; its group of fract. is Thompson's group F;
  - The category $\mathcal{A}^+$ is weakly left-Garside (and weakly right-G.).

- **Bad news: The Garside structure is trivial.**
  - The category $\mathcal{A}^+$ is not strongly left-Garside, and not Garside.
  - ϕ is constant on terms of a given size: $\phi(t) =$  for $\mathrm{size}(t) = 4$.
  - Every morphism is simple.

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