Surface invariants of finite type

INSTITUT FOURIER

Michael Eisermann
www-fourier.ujf-grenoble.fr/˜eiserm

TRESSES GDR 2105 CNRS

Summary

Motivation

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- Common framework
- Beautiful structure

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First modest results indicate that this is successful.

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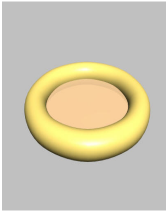
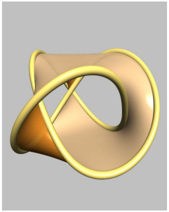
Embedded and immersed surfaces in $\mathbb{R}^3$

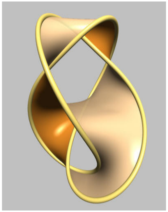
Embedded surfaces bounding knots or links:



(a) trivial knot, $\bigcirc$



(b) trefoil knot, 3_1



(c) figure eight, 4_1

SeifertView, Jarke van Wijk, TUE

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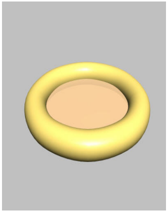

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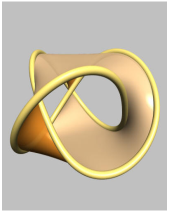

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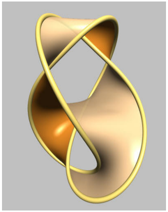

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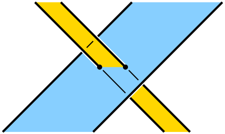
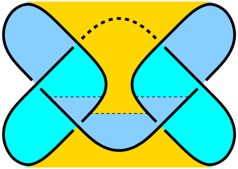
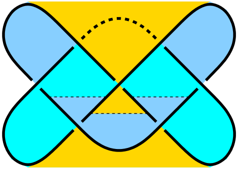
Immersed surfaces having only ribbon singularities:



(d) ribbon singularity

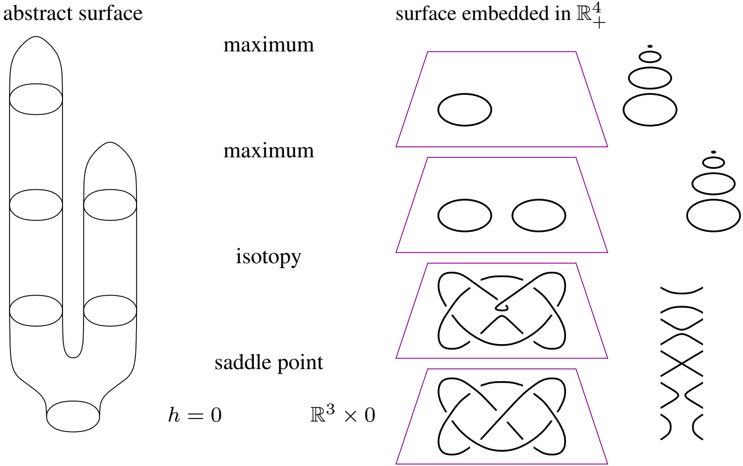


(e) $3_1 \sharp 3_1^*$



(f) 6_1

Relationship with surfaces in $\mathbb{R}^4_+$



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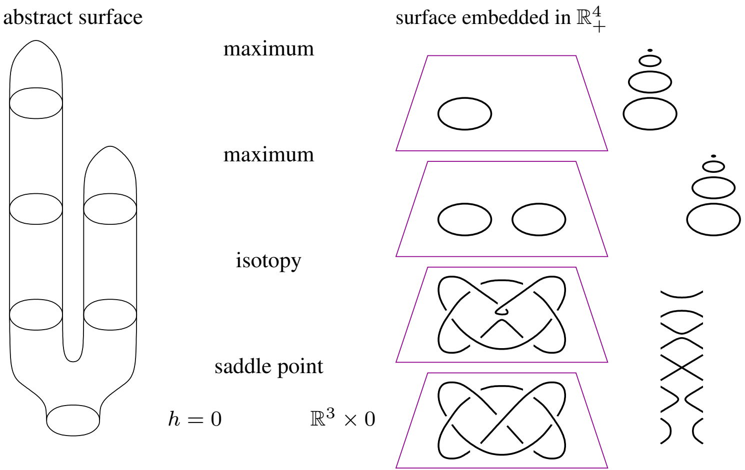

Proposition (Fox 1962)

A link $L \subset \mathbb{R}^3$ bounds an immersed ribbon surface $\Sigma \looparrowright \mathbb{R}^3$ iff it bounds a smoothly embedded surface $\Sigma \hookrightarrow \mathbb{R}^4_+$ without local minima.

Band diagrams

Let Σ be a compact oriented surface without closed components.

Definition

A *ribbon immersion* $F\colon \Sigma \looparrowright \mathbb{R}^3$ has only ribbon singularities.
A *ribbon surface* $S = F(\Sigma)$ is the image of a ribbon immersion F.

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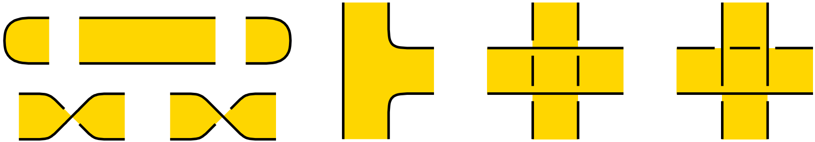

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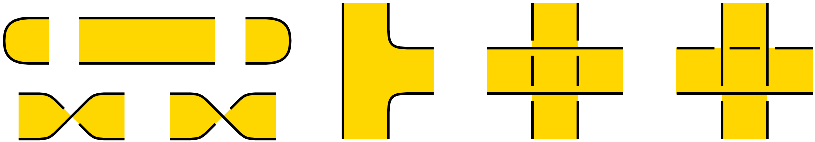
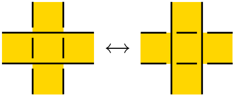


Proposition

Every ribbon surface S in $\mathbb{R}^3$ can be presented by a band diagram.

Band crossing changes

We consider *band crossing changes:*



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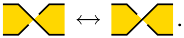

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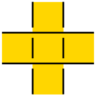
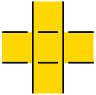
 ↔  or  ↔  .

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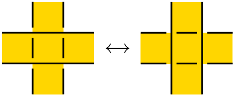

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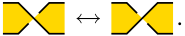

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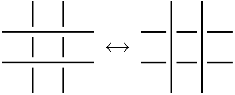
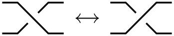
Forgetting the surface would lead to a coarser theory:



Kauffman's band pass moves

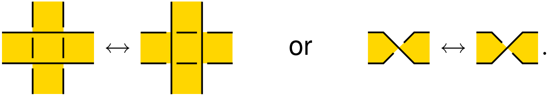


ordinary crossing changes

Surface invariants of finite type

Let D be a band diagram and let X be a set of band crossings.

Given $Y \subset X$ we obtain D_Y by changing the crossings in Y:



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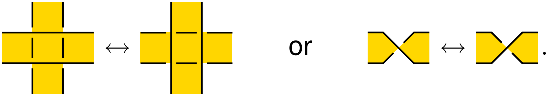

Definition

An invariant $v\colon \mathscr{S} \to A$ is of degree $\leq m$ if

$$\sum_{Y \subset X} (-1)^{|Y|} v(D_Y) = 0 \qquad \text{for all } X \text{ with } |X| > m.$$

We say that v is of *finite type* if v is of degree $\leq m$ for some $m \in \mathbb{N}$.

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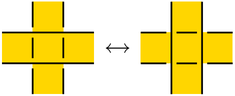

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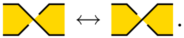

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We say that v is of *finite type* if v is of degree $\leq m$ for some $m \in \mathbb{N}$.

v is of degree $< 0 \iff v = 0$,

v is of degree $\leq 0 \iff v$ is constant,

v is of degree $\leq 1 \iff v$ is “at most linear”,

v is of degree $\leq 2 \iff v$ is “at most quadratic”, etc.

Invariants of finite type

Example

The Euler characteristic $S \mapsto \chi(\Sigma)$ is a surface invariant of degree 0.

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Proposition

If $\mathscr{L} \xrightarrow{v} A$, $L \mapsto v(L)$, *is a link invariant of degree* $\leq m$, *then*
$\mathscr{S} \xrightarrow{\partial} \mathscr{L} \xrightarrow{v} A$, $S \mapsto v(\partial S)$, *is a surface invariant of degree* $\leq m$.

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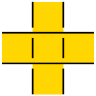
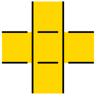
Proof. If we forget the surfaces in the band crossings

 $-$  and  $-$ ,

then we obtain (telescopic sums of) crossing changes of links. □

Seifert matrix and determinant

Assume the surface Σ to be compact, oriented and connected.

We have $\chi(\Sigma) = 1 - \operatorname{rk} H_1(\Sigma)$ because $H_0(\Sigma) = 1$ and $H_2(\Sigma) = 0$.

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To each embedding $F \colon \Sigma \hookrightarrow \mathbb{R}^3$ we associate its Seifert form

$$\theta_F \colon H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}, \qquad \theta_F(a,b) = \operatorname{lk}(F^{\uparrow}(a), F^{\downarrow}(b)).$$

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The coefficients of θ_F are of degree ≤ 1.

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The determinant of F is defined by $\det(F) := \det\big[-i(\theta_F + \theta_F^*)\big]$.

It is a homogeneous polynomial of degree m in the coefficients of θ_F.

Conclusion

The surface invariant $F \mapsto \det(F)$ is of degree $\leq m = 1 - \chi(\Sigma)$.

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⚠ The invariant $\det(F)$ depends only on the link $L = F(\partial\Sigma)$, but $L \mapsto \det(L)$ is not of finite type in the sense of Vassiliev–Goussarov.

Alexander polynomial

The same arguments hold for $\Delta(F) = \det(q^- \theta_F^* - q^+ \theta_F)$.

(We recover the determinant $\det(F) = \Delta(F)_{q \mapsto i}$ as a specialization.)

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Theorem (Seifert 1934)

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Theorem (Seifert 1934)

The invariant $\Delta(F)$ *depends only on the link* $L = F(\partial\Sigma)$*:*

Question (cf. Murakami–Ohtsuki 2001)

Which polynomials in θ_F are invariants of $L = F(\partial\Sigma)$?

Summary

Skein relations

Theorem (HOMFLY-PT)

For each $N \in \mathbb{N}$ there exists a unique invariant $V_N : \mathscr{L} \to \mathbb{Z}[q^{\pm}]$ satisfying $V_N(\bigcirc) = 1$ and the skein relation

$$q^{-N} \cdot V_N\left(\diagup\!\!\!\!\diagdown\right) - q^{+N} \cdot V_N\left(\diagdown\!\!\!\!\diagup\right) = (q^{-1} - q^{+1}) \cdot V_N\left(\rangle\,\langle\right).$$

$N = 0$: Alexander 1928, Conway 1969
$N = 1$: trivial invariant, $V_1 = 1$
$N = 2$: Jones 1984, $V := V_2$
$N > 2$: HOMFLY-PT 1985-1987

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Remark

$V(\bigcirc^n) = (q^{-1} + q^{+1})^{n-1}$ and $q^{-1} + q^{+1}$ is the minimal polynomial of i.

The situation is similar for V_N if N is prime.

Kauffman's bracket

Definition

There exists a unique map $\mathscr{D} \to \mathbb{Z}[A^{\pm}]$, denoted $D \mapsto \langle D \rangle$, such that

$$\langle \bigcirc \rangle = 1,$$

$$\langle D \sqcup \bigcirc \rangle = \langle D \rangle \cdot (-A^{+2} - A^{-2}),$$

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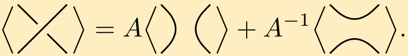

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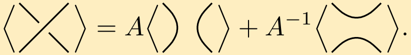

Theorem (Kauffman 1987)

We have $V(L)|_{(q \mapsto -A^{-2})} = \langle D \rangle \cdot (-A^{-3})^{\mathrm{writhe}(D)}$.

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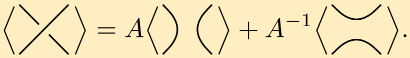

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Similar construction for V_N by Murakami–Ohtsuki–Yamada 1998

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This corresponds to the Seifert nullity $\mathrm{null}(L) = \mathrm{null}(\theta + \theta^*)$.

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Partial answer:

Theorem (E 2007)

For every n-component ribbon link we have $\operatorname{null} V(L) = n - 1$.

Jones nullity (lower bound)

Proposition (E 2007)

If a link $L \subset \mathbb{R}^3$ bounds a ribbon surface $S \subset \mathbb{R}^3$ of positive Euler characteristic n, then $V(L)$ is divisible by $V(\bigcirc^n) = (q^+ + q^-)^{n-1}$.

More succinctly: $V(\partial S)$ is divisible by $(q^+ + q^-)^{\chi(S)-1}$.

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Induction on the number $r(S)$ of ribbon singularities.

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Thus $\chi(S_i) > 0 \;\Leftrightarrow\; S_i = \bigcirc \;\Leftrightarrow\; \chi(S_i) = 1$.

Induction on the number $r(S)$ of ribbon singularities.

If $r(S) = 0$ then S is embedded and $L = L_0 \sqcup \bigcirc^n$.

Jones nullity (lower bound)

Proposition (E 2007)

If a link $L \subset \mathbb{R}^3$ bounds a ribbon surface $S \subset \mathbb{R}^3$ of positive Euler characteristic n, then $V(L)$ is divisible by $V(\bigcirc^n) = (q^+ + q^-)^{n-1}$.

More succinctly: $V(\partial S)$ is divisible by $(q^+ + q^-)^{\chi(S)-1}$.

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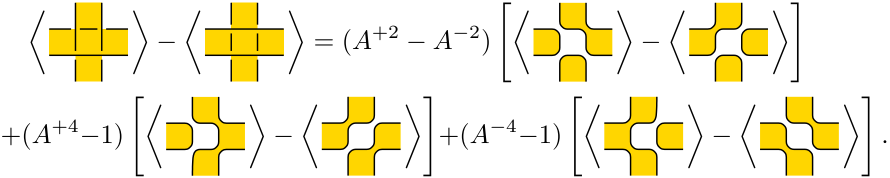
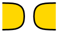
If $r(S) \geq 1$ then we consider the Kauffman bracket:



We conclude by induction using $\chi($$) = \chi($$) + 1$. □

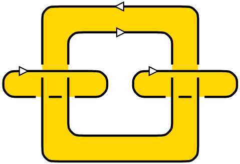
Jones nullity (examples)

Example

We have $\chi(S) = 1 + 1 + 0 = 2$ and

$$V(L) = (q^{+} + q^{-}) \cdot \left(q^{6} - q^{4} + 2q^{2} + 2q^{-2} - q^{-4} + q^{-6}\right).$$

Hence L bounds surfaces with $\chi \leq 2$ but not with $\chi \geq 3$.

$S =$ 

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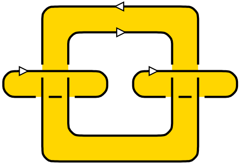
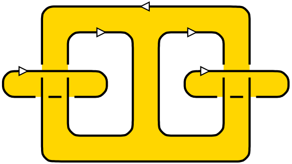
$S =$ 

$S' =$ 

Example

We have $\chi(S') = 1 + 1 - 1 = 1$. Notice that $L' = \partial S'$ is the connected sum $H_{+} \sharp H_{-} \sharp H_{+} \sharp H_{-}$ of Hopf links, whence

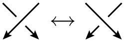
$$V(L) = (q^{+1} + q^{+5})^{2} \cdot (q^{-1} + q^{-5})^{2}.$$

Thus L' bounds surfaces with $\chi \leq 1$ but not with $\chi \geq 2$.

Expansion into finite type invariants

Expand $V(L) = \sum_{k=0}^{\infty} v_k(L) \cdot h^k$ in $q = \exp(h/2)$.

Then $L \mapsto v_k(L)$ is of degree $\leq k$ w.r.t. crossing changes:



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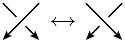

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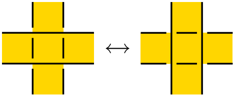

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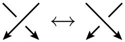

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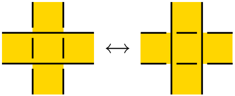

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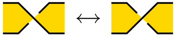

.

⚠ $d_k(L)$ is not of finite type in the sense of Vassiliev–Goussarov.
In particular $d_0(L) = V(L)_{q \mapsto i} = \Delta(L)_{q \mapsto i} = \det(L) = \det\left[-i(\theta + \theta^*)\right]$.

Inductive proof

Proposition (E 2007)

The surface invariant $S \mapsto d_k(\partial S)$ *is of degree* $\leq m := k + 1 - \chi(S)$.

The case $d_0 = \det$ has already been derived from the Seifert matrix.

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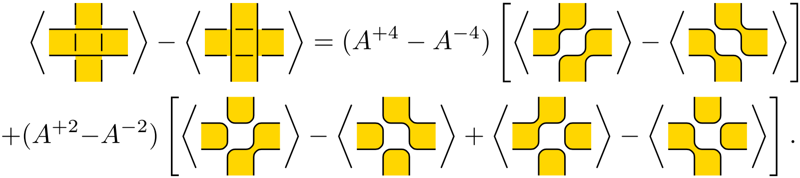
Moreover, $\sum_{Y \subset X} (-1)^{|Y|} V(\partial D_Y)$ is divisible by $(q^+ + q^-)^{|X|+\chi(S)-1}$:



We conclude by induction on $|X|$. □

Summary

Tangled surfaces

Consider the category generated by embedded surface pieces:

(a) id (b) twists (c) crossings (d) junctions (e) ends

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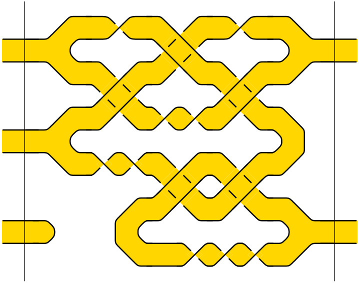

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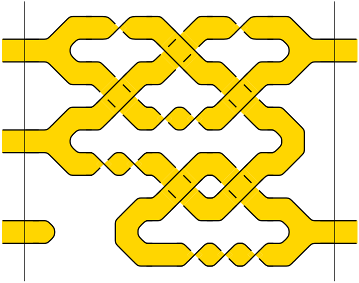
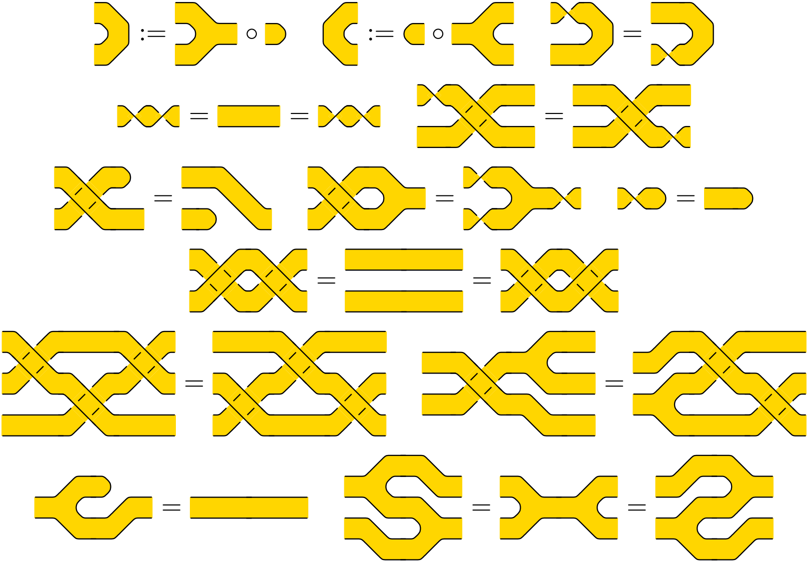


For ribbon immersions $\Sigma \looparrowright \mathbb{R}^3$ the construction is similar but longer.

Isotopy relations



Abstract surfaces

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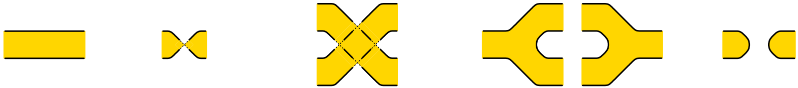

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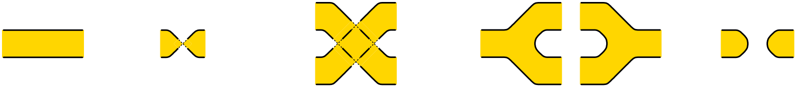

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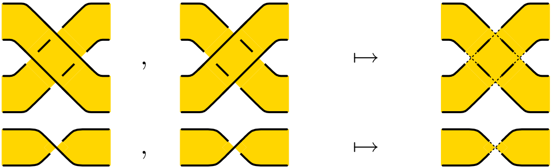

Chord diagrams on surfaces

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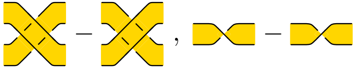
$$I = \Bigg($$  $$\Bigg)$$

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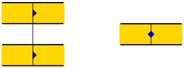

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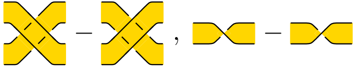
$I = \Big($  $\Big)$

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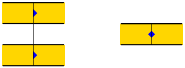

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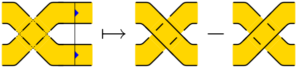

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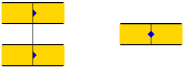

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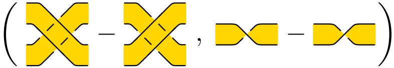

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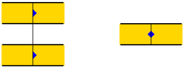

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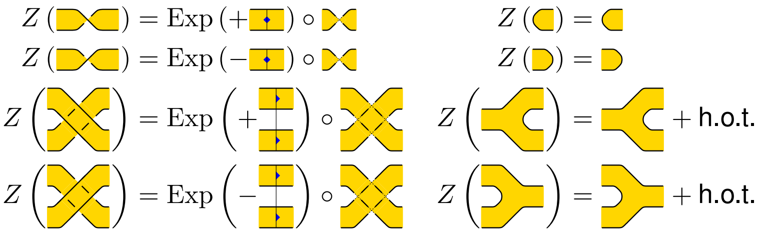
Question

Is the quotient finite dimensional in each degree?

Towards a universal invariant

We wish to define a universal invariant Z as follows:



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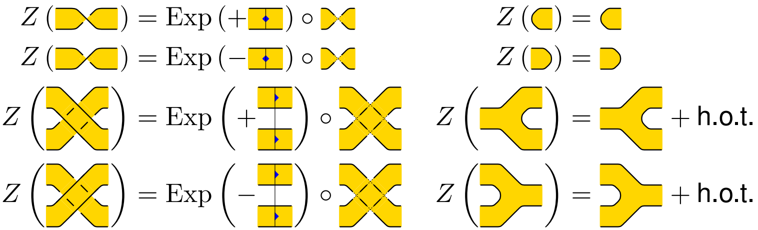

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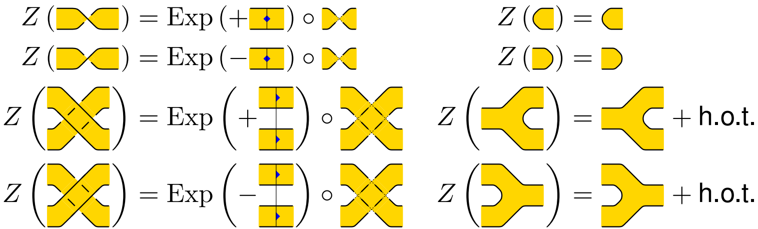

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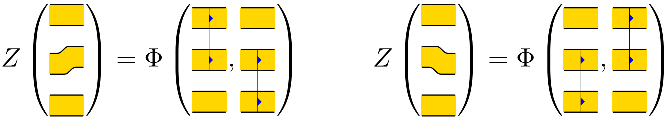

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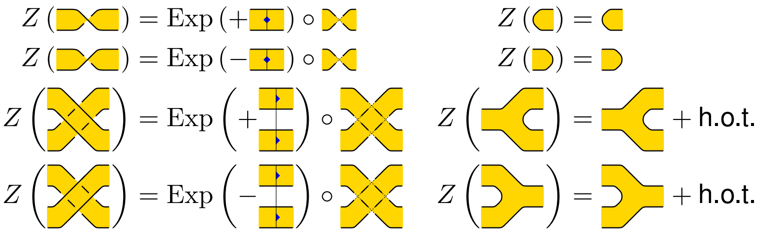

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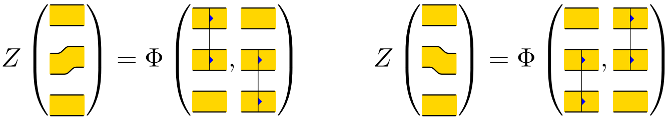

Question

Do all isotopy relations hold? Can we arrange this?

Open questions

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Thank you for your attention.