Twisted conjugacy in braid groups

Juan González-Meneses

Universidad de Sevilla

Joint with **E. Ventura**.

Tresses à Paris

Rencontres Parisiennes du GDR Tresses

Paris, 17-20 september 2008.

In a group G:

Conjugation: $c^{-1}\, a\, c = b$ $\boxed{a \xrightarrow{c} b}$ $\boxed{a \sim b}$

Conjugacy Decision Problem:

Determine whether two elements are conjugate.

Conjugacy Search Problem:

Find a conjugating element for two given conjugate elements.

In a group G: Reidemeister (1936)

Twisted Conjugation: $f(c)^{-1}\,a\,c = b$ $f:\ G \to G$

Fixed automorphism.

Twisted Conjugacy Decision Problem:

Determine whether two elements are twisted conjugate.

Twisted Conjugacy Search Problem:

Find a conjugating element for two given twisted conjugate elements.

Introduction

If $f : G \to G$ is an inner automorphism: $f(x) = \alpha^{-1} x \alpha$

Twisted conjugacy problem $\boxed{\exists c, \quad f(c)^{-1} a c = b}$

$$\boxed{\exists c, \quad \alpha^{-1} c^{-1} \alpha a c = b}$$

Conjugacy problem $\boxed{\exists c, \quad c^{-1} (\alpha a) c = (\alpha b)}$

Just need to focus on representatives of $Out(G) = Aut(G)/Inn(G)$.

Bogopolski, Martino, Ventura, 2008.

$$1 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 1$$

Solvable twisted conj. problem. ?

Solvable conj. problem.
& $[C_H(h) : \langle h \rangle] < \infty \quad \forall h \in H.$

$F =$ f.g.-abelian | f.g.-free | ...

$H =$ f.g.-free | f.g.-t.f.-hyperbolic | ...

G has solvable conj. problem $\Leftrightarrow$ $A_G < Aut(F)$ is **orbit decidable**

One can determine, given $x, y \in F$, whether $x \sim f(y)$, for some $f \in A_G$.

Can we put braid groups here?

✓ ($Out B_n$ is finite)

Braid groups

B_n: Braid group on n strands.

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i, & |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & 1 \leq i \leq n-2 \end{array} \right\rangle$$

Left normal form: $x = \Delta^p x_1 \cdots x_r$

Each factor is a **simple element**. (**permutation braid**)

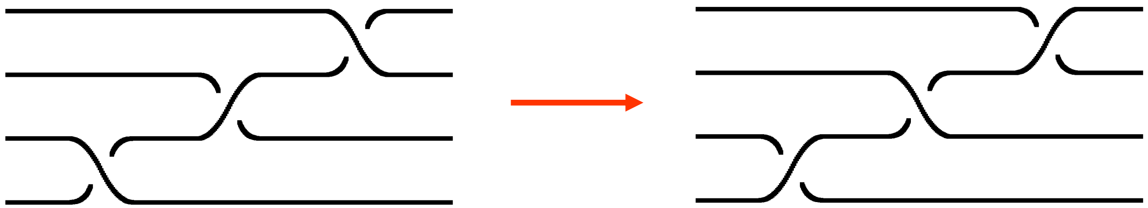
Canonical length = No. of factors. $\ell(x) = r$.

Automorphisms of B_n: (Dyer-Grossman, 1981)

$$Out(B_n) \;=\; Aut(B_n)/Inn(B_n) \;=\; \{\mathrm{id},\, \varepsilon\}$$

$$\begin{aligned} \varepsilon: \; B_n &\longrightarrow B_n \\ \sigma_i &\longmapsto \sigma_i^{-1} \end{aligned}$$

$$\varepsilon(\;\sigma_1\,\sigma_2^{-1}\sigma_3\;) \;=\; \sigma_1^{-1}\sigma_2\,\sigma_3^{-1}.$$



Just need to solve the twisted conjugacy problem for ε.

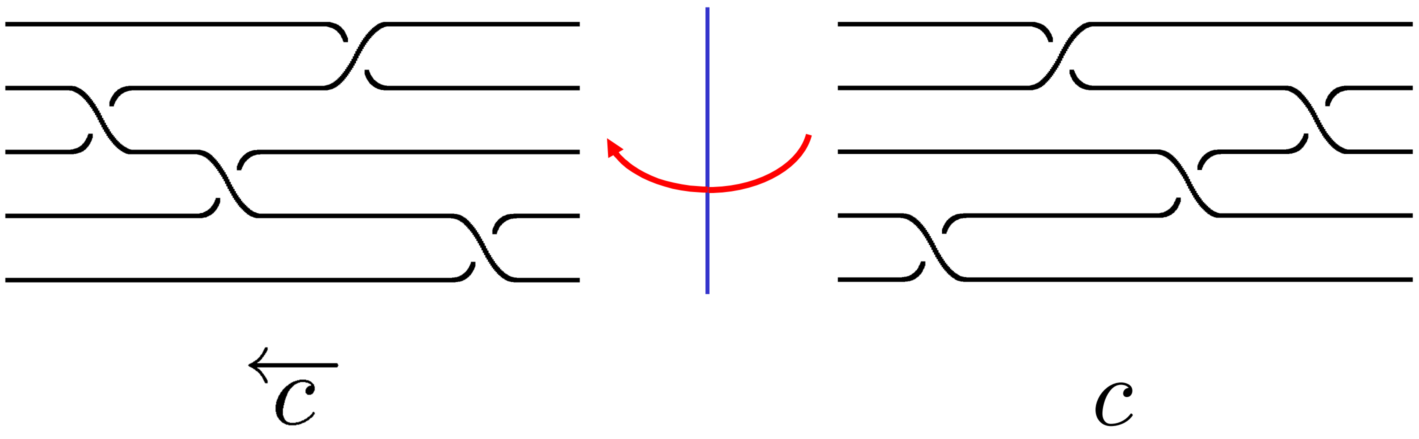
Twisted conjugation for ε: $\varepsilon(c)^{-1}\, a\, c$

$$c = \sigma_1\sigma_4^{-1}\sigma_2\sigma_3$$

$$\varepsilon(c) = \sigma_1^{-1}\sigma_4\sigma_2^{-1}\sigma_3^{-1}$$

$$\varepsilon(c)^{-1} = \sigma_3\sigma_2\sigma_4^{-1}\sigma_1 = \overleftarrow{c}$$

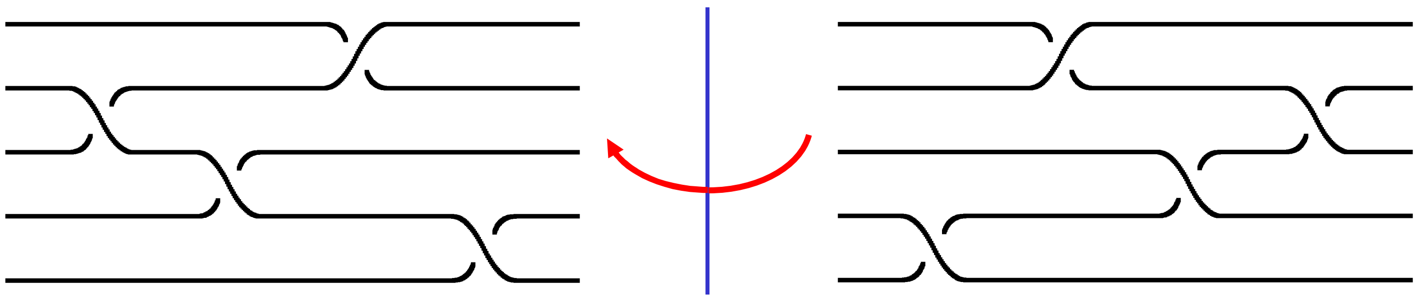
(c written backwards)



Twisted conjugation for ε: $\varepsilon(c)^{-1}\, a\, c$

$$\boxed{\varepsilon(c)^{-1}\, a\, c \;=\; \overleftarrow{c}\; a\, c}$$

This is the twisted conjugation we will consider.



$\overleftarrow{c}$ c

$$\sigma_1 \xrightarrow{\text{Twst conj}} \sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1 \xrightarrow{\text{Twst conj}} \sigma_2$$

$\sigma_1, \ldots, \sigma_{n-1}$ are **twisted conjugate**.

$\sigma_1\sigma_2, \quad \sigma_2\sigma_1$ are **conjugate**, but not **twisted conjugate**.

How to solve the twisted conjugacy problems?

Back to the conjugacy problem

Algorithm to solve the conjugacy problem. (ElRifai-Morton, 1988)

Compute a finite set, invariant of the conjugacy class.

$$\text{SSS}(x) = \Big\{ \text{conjugates of x, of minimal canonical length} \Big\}$$

One can compute SSS(x) using the following:

Theorem (Elrifai-Morton, 1988)**:** Let $u,v \in B_n$ conjugate, $\ell(u), \ell(v) \leq r$. Then u and v can be joined through conjugations by simple elements, where every intermediate conjugate w has $\ell(w) \leq r$.

Back to the conjugacy problem

Theorem (Elrifai-Morton, 1988)**:** Let $u, v \in B_n$ conjugate, $\ell(u), \ell(v) \leq r$. Then u and v can be joined through conjugations by simple elements, where every intermediate conjugate w has $\ell(w) \leq r$.

(can assume positive)

$$u \xrightarrow[c_1 c_2 \cdots c_r]{c} v$$

(left normal form)

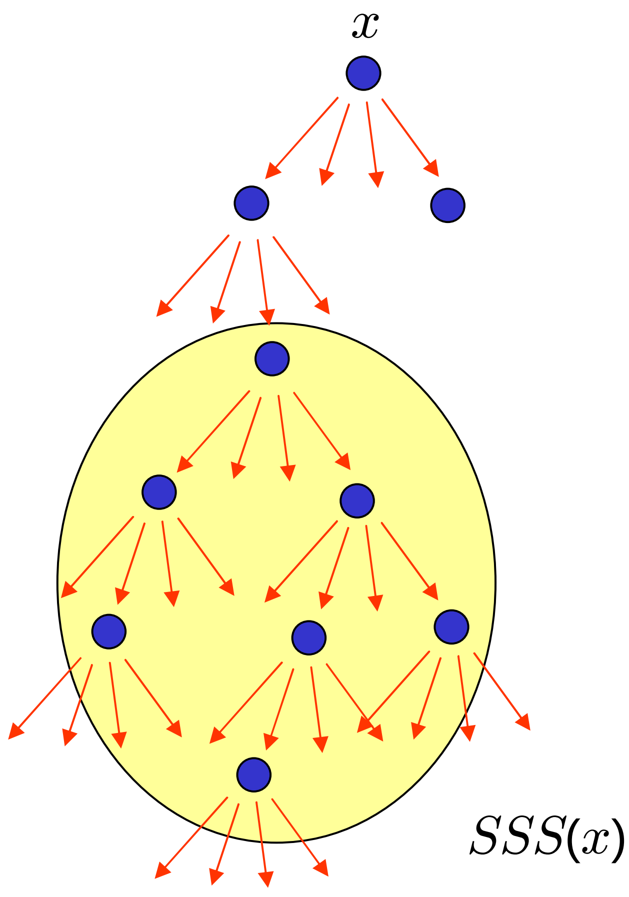
$$u \xrightarrow{c_1} w_1 \xrightarrow{c_2} w_2 \longrightarrow \ldots \longrightarrow w_{r\text{-}1} \xrightarrow{c_r} v$$

Each c_i is simple

$\ell(w_i) \leq r, \quad \forall i$

Back to the conjugacy problem

Computing $SSS(x)$:



Conjugate by all simple elemets...

…keeping elemets of minimal length.

If no new element is found, $SSS(x)$ is computed.

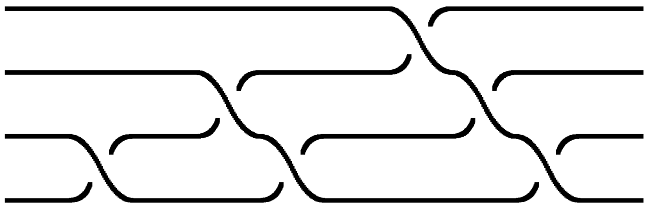
This solves the conjugacy problem.

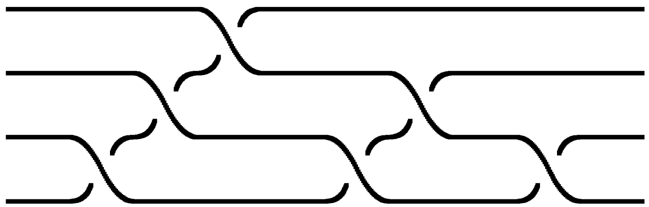
Twisted conjugacy problem

First idea: Restrict to **positive** braids.

For every $x \in B_n$, $x\Delta^p$ is positive for p big enough.

$\overleftarrow{\Delta} = \Delta$





$$x \xrightarrow{\text{Twst conj}} \overleftarrow{\Delta^p}\, x\, \Delta^p = \Delta^p\, x\, \Delta^p$$ **Positive!**

Every braid is twisted conjugate to a positive braid.

Twisted conjugacy problem

First idea: Restrict to **positive** braids.

Every braid is twisted conjugate to a positive braid.

But…

The set of positive twisted-conjugates of x is **infinite**.

$$\cdots\ \sigma_5\sigma_2\sigma_3\sigma_1\ \text{(braid)}\ \sigma_1\sigma_3\sigma_2\sigma_5\ \cdots$$

Twisted conjugacy problem

A positive braid x is **palindromic-free** if it cannot be written as:

$$x = \sigma_i \, y \, \sigma_i$$

Second idea: Restrict to **positive**, **palindromic-free** braids.

Every positive braid is twisted conjugate to a palindromic-free one.

$$\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1 \longrightarrow \sigma_2\sigma_3\sigma_2 \longrightarrow \sigma_3$$

$$\parallel$$

$$\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3 \longrightarrow \sigma_2\sigma_1\sigma_2 \longrightarrow \sigma_1$$

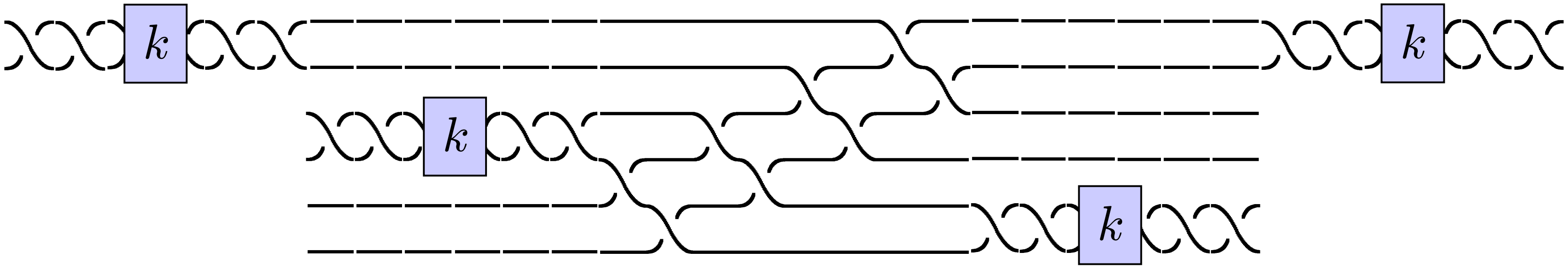
Second idea: Restrict to **positive**, **palindromic-free** braids.

Every positive braid is twisted conjugate to a palindromic-free one.

But…

The set of positive, palindromic-free twisted-conjugates of x can be infinite.

Example:



These braids are palindromic-free, for all k.

They are twisted conjugate.

Twisted conjugacy problem

Third idea: Restrict to **positive**, **palindromic-free** braids, of **minimal length**

$$MPF(x) = \left\{ \text{Twisted conjugates of x which are: } \begin{array}{r} \text{Positive,} \\ \text{Palindromic-free} \\ \text{of minimal length.} \end{array} \right\}$$

This is a finite set, invariant of the twisted-conjugacy class.

Computing $MPF(x)$, we solve the twisted.conjugacy problem.

How to compute it?

Computing *MPF*(x)

For usual conjugacy problem…

$$u \xrightarrow{c_1} w_1 \xrightarrow{c_2} w_2 \longrightarrow \cdots \longrightarrow w_{r\text{-}1} \xrightarrow{c_r} v$$

…simple conjugations.

For twisted conjugacy problem:

$$\sigma_1 \xrightarrow{\sigma_2} \sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1 \xleftarrow{\sigma_1} \sigma_2$$

$$\sigma_1 \xrightarrow{\sigma_2} \cdot \xleftarrow{\sigma_1} \sigma_2$$

Simple twisted-conjugation: $x \xrightarrow{\textit{simple}} \cdot \xleftarrow{\textit{simple}} y$

Computing *MPF*(x)

Theorem (Elrifai-Morton, 1988)**:** Let $u, v \in B_n$ conjugate, $\ell(u), \ell(v) \le r$.
Then u and v can be joined through conjugations by simple elements,
where every intermediate conjugate w has $\ell(w) \le r$.

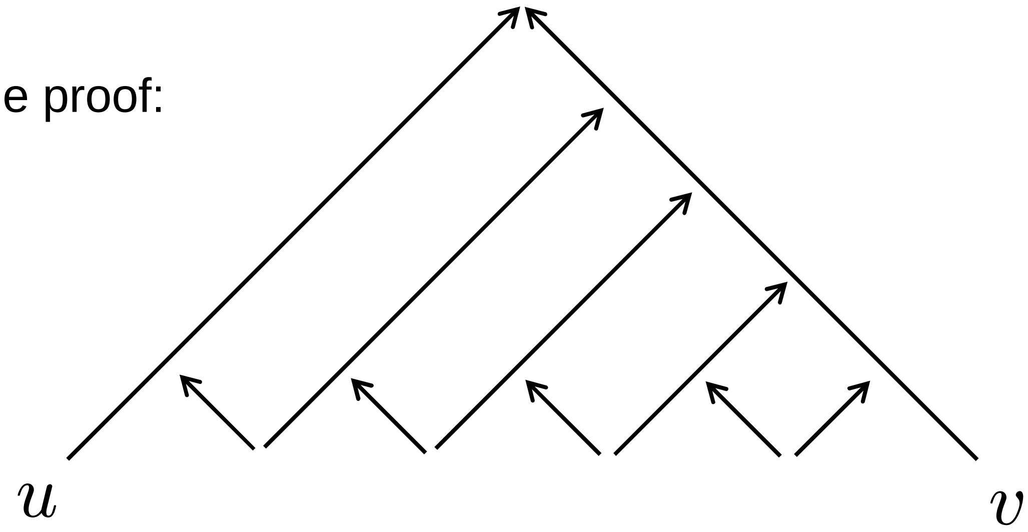
Theorem (GM-Ventura, 2008)**:** Let $u, v \in B_n$ twisted conjugate, $\ell(u), \ell(v) \le r$.
Then u and v can be joined through simple twisted-conjugations,
where every intermediate twisted-conjugate w has $\ell(w) \le r$.

$$u \qquad w_1 \qquad w_2 \qquad \dots \qquad w_{r-1} \qquad v$$

All palindromic-free

Computing *MPF*(x)

Ingredients of the proof:



$$\left.\begin{aligned} v &= \overleftarrow{c}\; u\; c \\ \varepsilon(v) &= \varepsilon(\overleftarrow{c}\; u\; c) \;=\; c^{-1}\,\varepsilon(u)\,\varepsilon(c) \end{aligned}\right\} \quad \varepsilon(v)\,v \;=\; c^{-1}\,\varepsilon(u)\cancelto{1}{\varepsilon(c)\,\overleftarrow{c}}\,u\;c$$

$$\varepsilon(v)\,v = c^{-1}\,\varepsilon(u)\;u\;c$$

If u and v are twisted conjugate, then $\varepsilon(u)u$ and $\varepsilon(v)v$ are conjugate.

Then use Elifai-Morton's Theorem.

Conclusion

$$1 \longrightarrow B_n \longrightarrow G \longrightarrow H \longrightarrow 1$$

$H =$ f.g.-free f.g.-t.f.-hyperbolic …

Since $A_G < Aut(B_n)$ is **orbit decidable**,

and B_n has solvable **twisted conjugacy problem**,

G has solvable conjugacy problem

(decision & search)