Braid monodromies and its applications in low dimensional topology

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I'd like to explain how braid monodromies appear in studies of various topological objects in low dimensional topology; braided surfaces, 2-dimensional braids, Lefschetz fibrations of 4-manifolds, etc. Such a braid monodromy can be expressed by a system of elements of the braid group or the mapping class group (in general, elements of quandles), and can be also expressed by some graphics called charts.

algebraic curves (stable branch curves)
(B. Moishezon '81, B.M. and M. Teicher)

braided surfaces
(L. Rudolph '82)

2-dimensional braids
(O. Viro and S. Kamada '92,
S. Carter, M. Saito, S. Satoh)

Lefschetz fibrations
(Y. Matsumoto '86 for topological approach, etc)

$$\Longrightarrow \frac{\left\{ \begin{array}{c} \text{braid monodromy} \\ \text{representations} \end{array} \right\}}{\sim}$$

$$\Longleftrightarrow \frac{\text{braid systems } (b_1, \ldots, b_n) \in G \times \cdots \times G}{\sim\text{: Hurwitz equivalence}} \quad \Longleftrightarrow \frac{\text{graphics (called charts)}}{\sim}$$

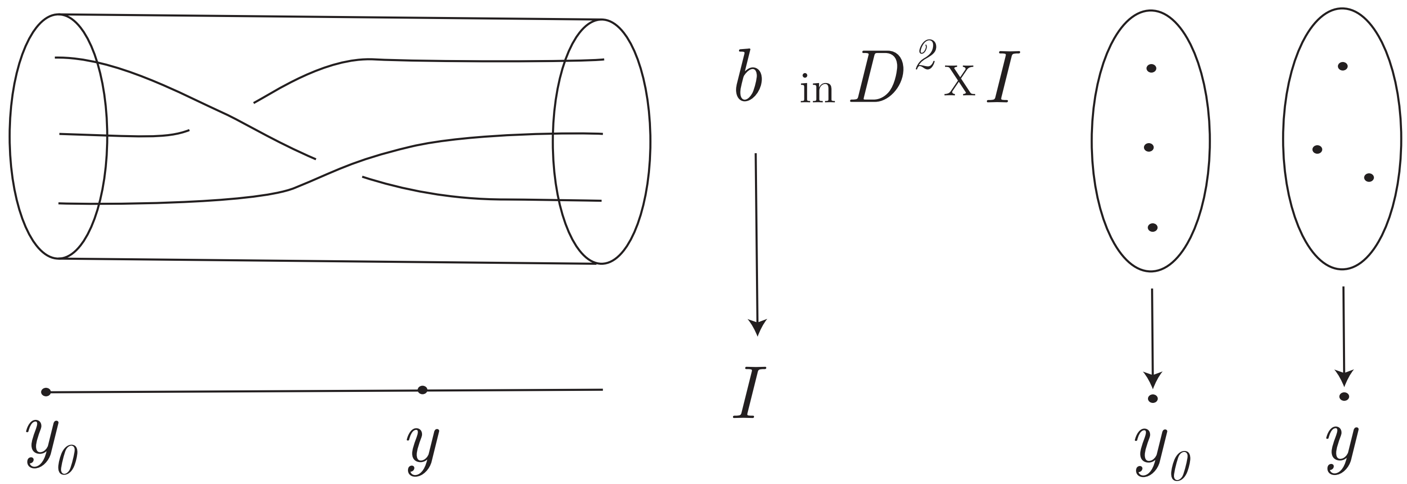
where G is a braid group, a mapping class group, or a subgroup of them.

Let b be a 1-manifold in $D^2 \times I$.

b is a braid if (1A) and (1B) are satisfied.

(1A) $pr_2|_b : b \to I$ is a covering map of degree m.

(1B) For every $y \in \partial I$, $(pr_2|_b)^{-1}(y)$ is a fixed m points of D^2.



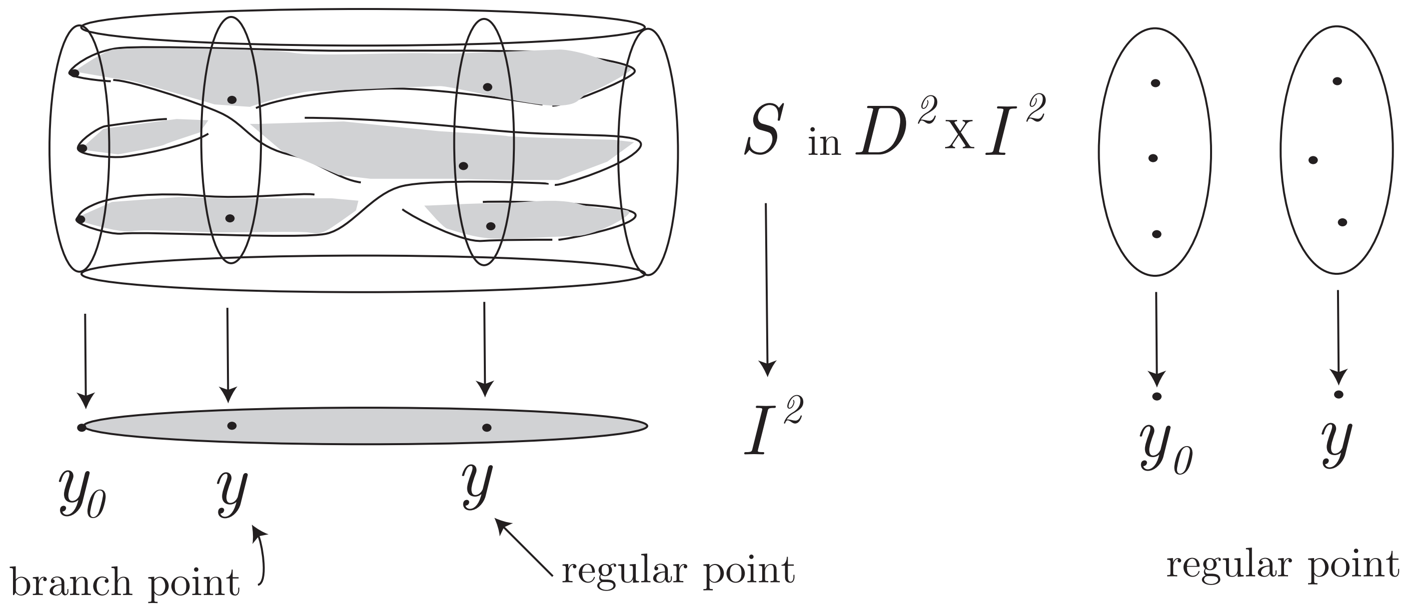
Let S be a 2-manifold S in $D^2 \times I^2$.

S is a <u>2-dimensional braid</u> if (2A) and (2B) are satisfied.

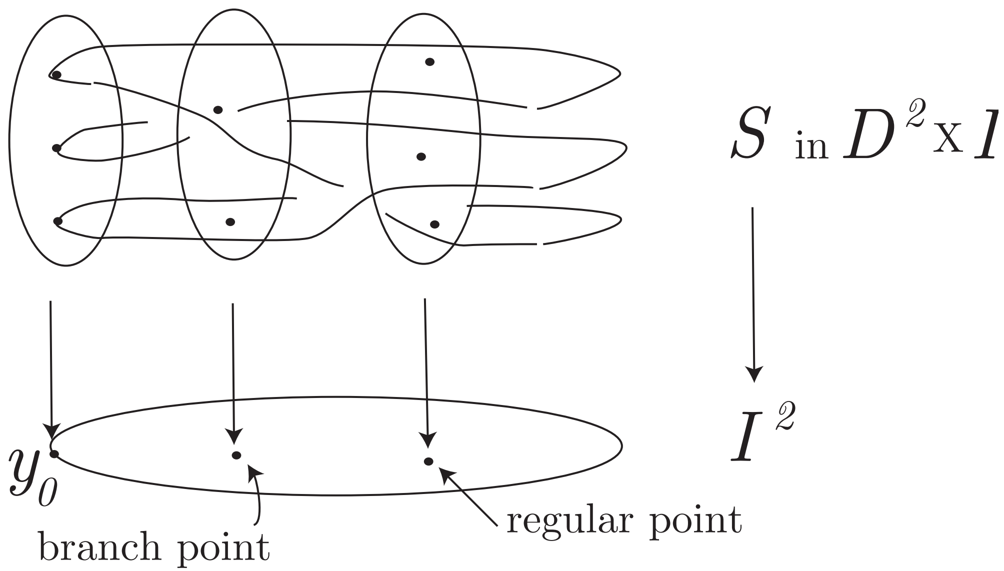
(2A) $pr_2|_S : S \to I^2$ is a branched covering map of degree m.

(2B) For every $y \in \partial I^2$, $(pr_2|_S)^{-1}(y)$ is a fixed m points of D^2.

(S is called a <u>braided surface</u> if (2A) is satisfied.)



For any regular point $y \in I^2$, the preimage $(pr_2|_S)^{-1}(y)$ is a set of m points of D^2. Thus we have a homomorphism $\rho : \pi_1(I^2 \setminus \Delta, y_0) \to B_m$, the braid monodromy representation of S, where $\Delta \subset I^2$ is the set of branch points for S.



Two monodromy representations $\rho : \pi_1(I^2 \setminus \Delta, y_0) \to B_m$ and $\rho' : \pi_1(I^2 \setminus \Delta', y_0) \to B_m$ are <u>equivalent</u> if there is a homeomorphism $h : I^2 \to I^2$ rel ∂I^2 such that $h(\Delta) = \Delta'$ and $\rho = \rho' \circ h_*$.

<u>Fact</u>

$$\{\text{2-dim braids}\} / \sim \overset{1:1}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{braid monodromy representations} \\ \text{of 2-dim braids} \end{array} \right\} / \sim$$

<u>Problems</u>

(1) Decide if a given map $\rho : \pi_1(I^2 \setminus \Delta) \to B_m$ is a braid monodromy representation of a 2-dim braid or not.

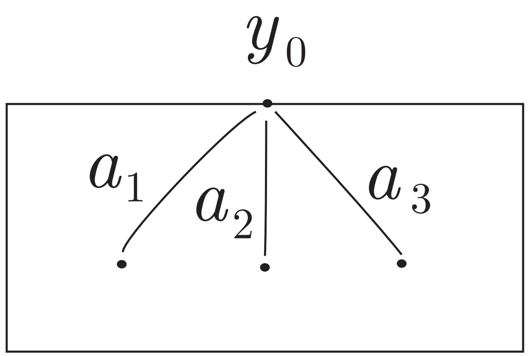
(2) Decide if two given monodromy representations are equivalent or not.

We shall restate these in terms of "braid systems".

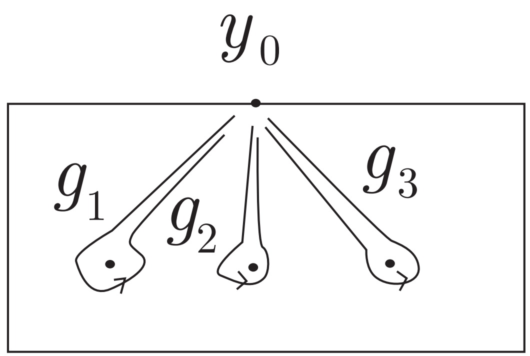
Let S be a 2-dim braid, $\rho : \pi_1(I^2 \backslash \Delta, y_0) \to B_m$ the braid monodromy representation.

Let $\Delta = \{y_1, \cdots, y_n\}$ $(\subset I^2)$.

Take a 'Hurwitz generator system', $\mathcal{G} = (g_1, \ldots, g_n)$, of $\pi_1(I^2 \setminus \Delta, y_0)$



Hurwitz arc system



Hurwitz generator system

The n-tuple $(\rho(g_1), \ldots, \rho(g_n))$ is a braid system of S (or of ρ). (It is not unique. It depends on a Hurwitz generator system $\mathcal{G} = (g_1, \ldots, g_n)$)

In general, the braid group B_n acts on the n-fold Cartesian product $P_n(G) = G \times \cdots \times G$ of a group G by the 'Hurwitz action' generated by the moves:

$$\sigma_i : (x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{i+1}, {x_{i+1}}^{-1} x_i x_{i+1}, \ldots, x_n)$$

$$\sigma_i^{-1} : (x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_i x_{i+1} x_i^{-1}, x_i, \ldots, x_n)$$

It is well known that two braid systems are in the same orbit if and only if their monodromy representations are equivalent.

$$\{\text{2-dim braids}\} / \sim \quad \overset{1:1}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{braid monodromy representations} \\ \text{of 2-dim braids} \end{array} \right\} / \sim$$

$$\overset{1:1}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{braid systems} \\ \text{of 2-dim braids} \end{array} \right\} / \text{Hurwitz equivalence}$$

Problems (stated before)

(1) Decide if a given map $\rho : \pi_1(I^2 \setminus \Delta) \to B_m$ is a braid monodromy representation of a 2-dim braid or not.

(2) Decide if two given monodromy representations are equivalent or not.

Problems (in terms of braid systems)

(1) Decide if a given n-tuple $(b_1, \ldots, b_n) \in P_n(B_m) = B_m \times \cdots \times B_m$ is a braid system of a 2-dim braid or not.

(2) Decide if two given braid systems $(b_1, \ldots, b_n)$ and $(b'_1, \ldots, b'_n)$ are Hurwitz equivalent or not.

I'll give an answer to (1) in the next slide. I don't know the answer to (2). A chart description method, explained later, is helpful to (2).

Define a subset A_m of B_m as follows: $b \in A_m$ if and only if

(i) b is not $1 \in B_m$.

(ii) b is conjugate to a braid b' such that $b' = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_c$ (split union of braids)

(iii) the closure of each β_i $(1 \leq i \leq c)$ is an unknot in S^3.

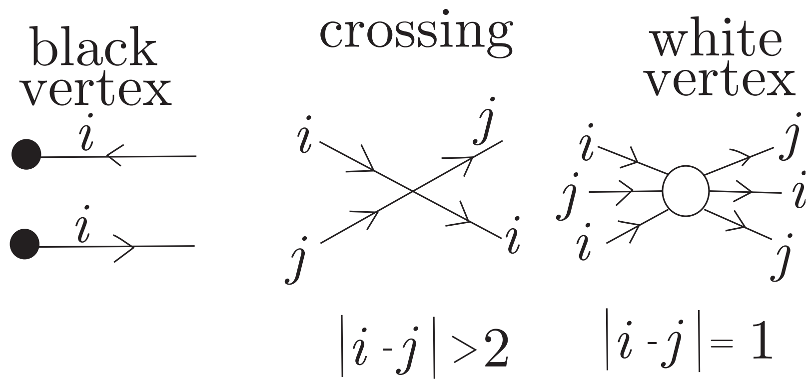
Theorem (K.'96) An n-tuple $(b_1, \ldots, b_n) \in P_n(B_m) = B_m \times \cdots \times B_m$ is a braid system of a 2-dim braid if and only if every $b_i \in A_m$, and the product $b_1 \cdots b_n$ is $1 \in B_m$.

Chart description: It is a method to describe 2-dimensional braids by graphics, called charts. For simplicity, we consider chart descriptions of 'simple' 2-dimensional braids. (A 2-dim braid is called simple if the branched covering $pr_2|_S : S \to I^2$ is simple. In other words, each element b_i of a braid system $(b_1, \ldots, b_n)$ is a conjugate of σ_1 or σ_1^{-1}.)

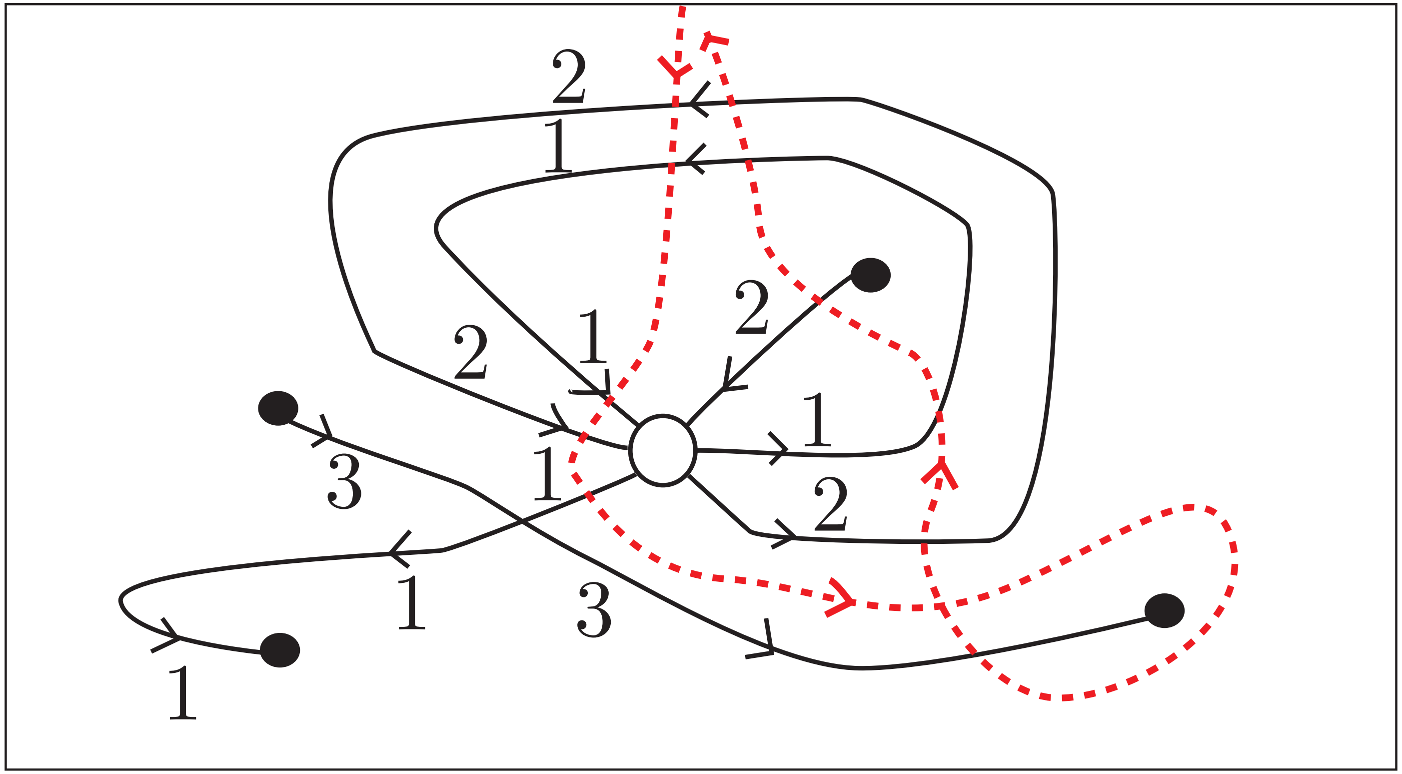
A chart is a graph Γ in I^2 satisfying

(i) each edge is oriented and labeled by an integer in $\{1, \ldots, m-1\}$,

(ii) each vertex is one of the following.

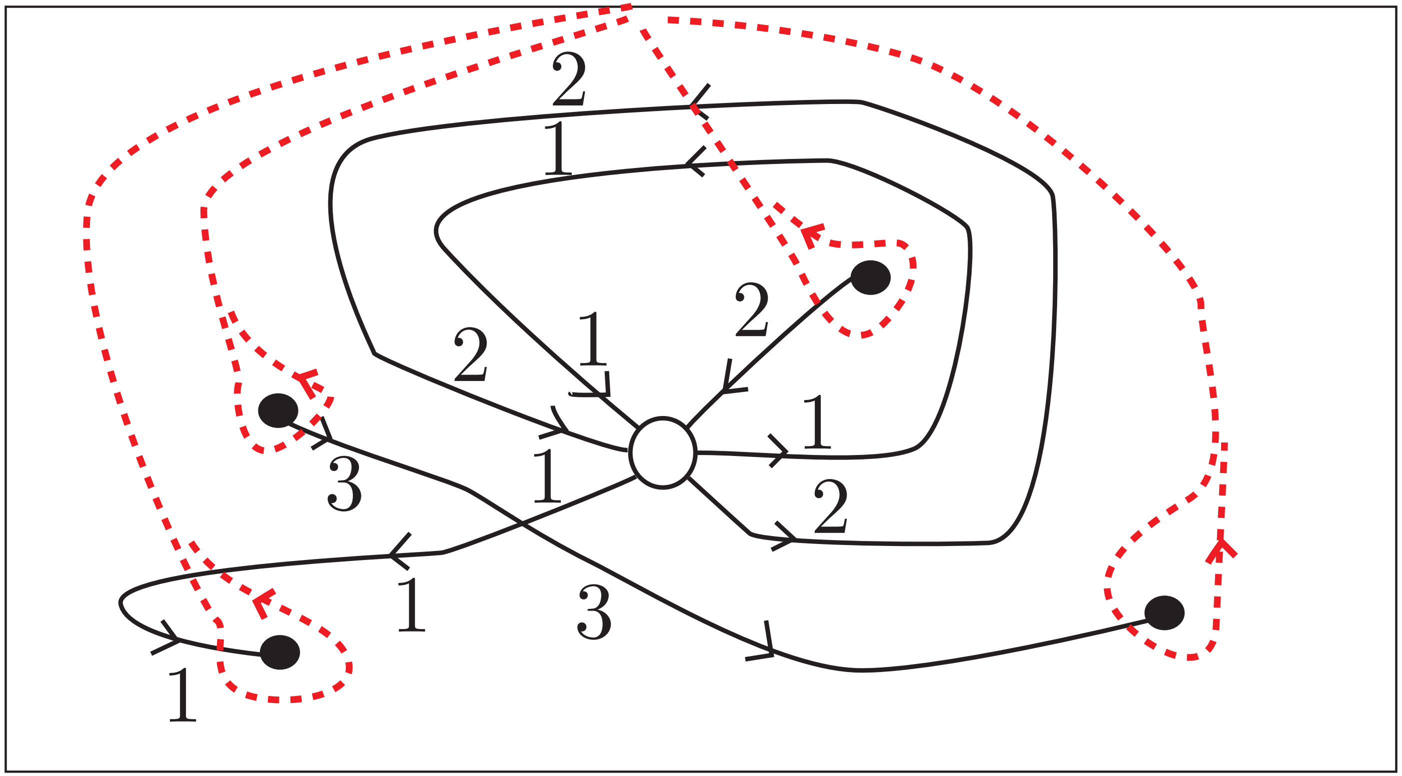


A chart Γ determines a braid monodromy $\rho_\Gamma : \pi_1(I^2 \setminus \Delta, y_0) \to B_m$, where Δ is the set of black vertices. (For a path, just read the 'intersection word'.)



$$\rho(c) = \sigma_2^{-1}\sigma_1^{-1}\sigma_1\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2$$

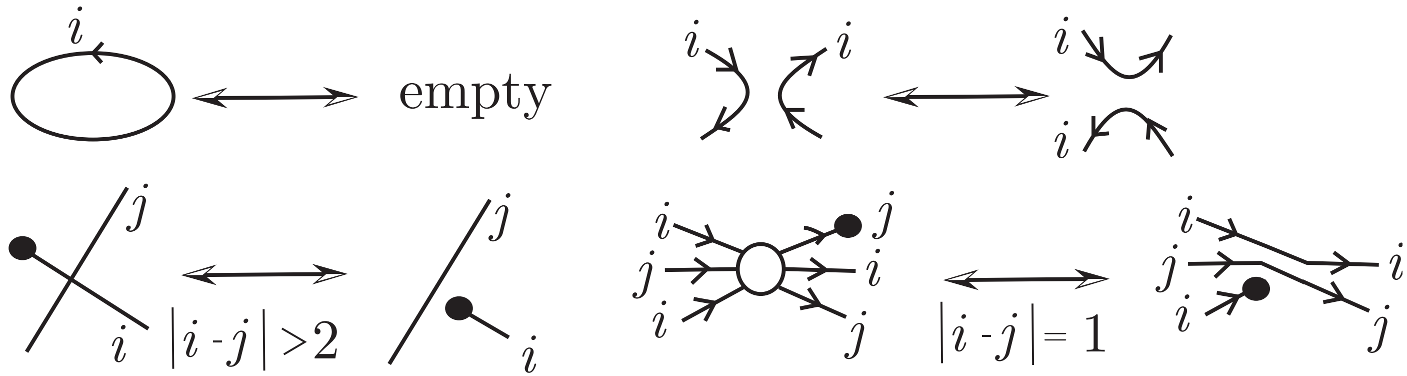
In order to obtain a braid system $(b_1, \ldots, b_n)$ from a chart, read the 'intersection words' along a Hurwitz generator system.



$$(b_1, b_2, b_3, b_4) = (\sigma_1^{-1}\sigma_1\sigma_1, \sigma_3, \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2, \sigma_3)$$

$$= (\sigma_1, \sigma_3, \sigma_1^{-1}, \sigma_3) \quad \left(\sim (\sigma_1, \sigma_1^{-1}, \sigma_3, \sigma_3^{-1})\right)$$

One can change charts by some basic moves, called <u>chart moves</u> without changing the monodromy representations.



$$\{\text{2-dim braids}\}/\sim \quad \overset{1:1}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{braid monodromy representations} \\ \text{of 2-dim braids} \end{array} \right\} / \sim$$

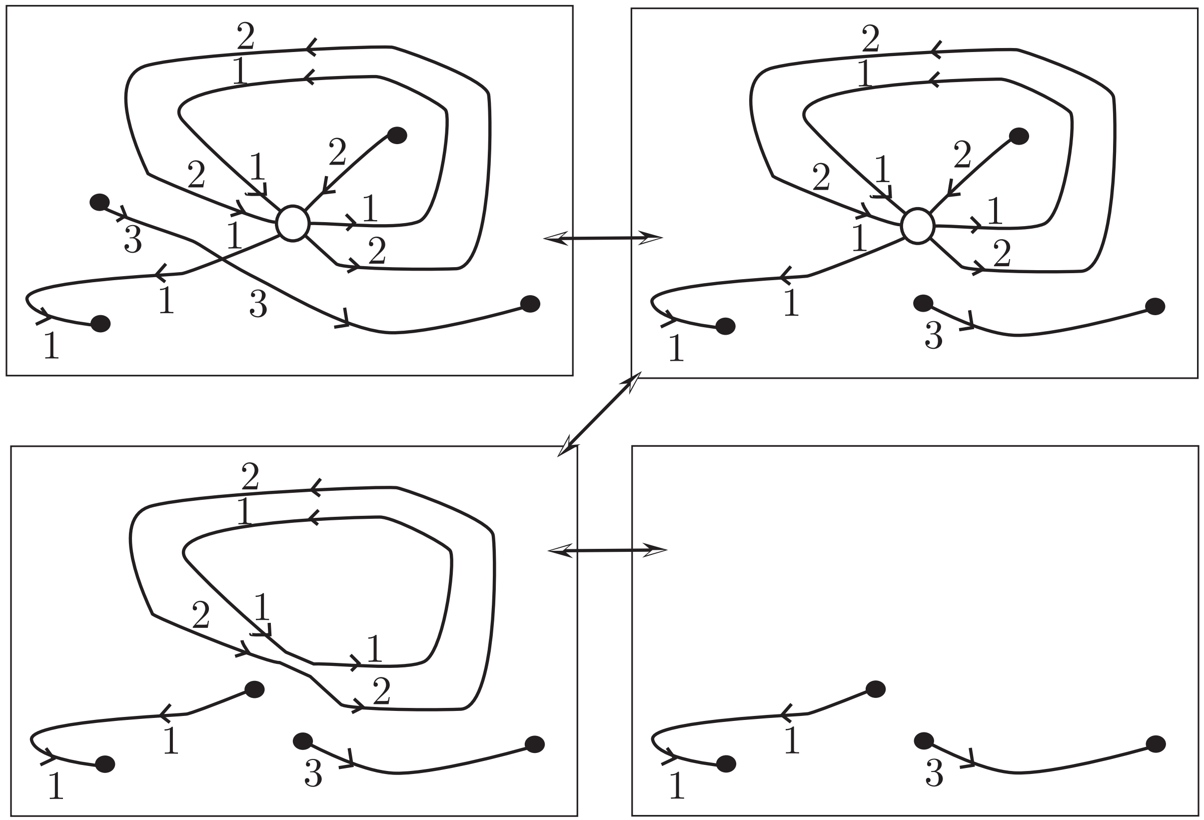
$$\overset{1:1}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{braid systems} \\ \text{of 2-dim braids} \end{array} \right\} /\text{Hurwitz equivalence}$$

$$\overset{1:1}{\Longleftrightarrow} \{\text{charts}\}/\text{chart move equivalence}$$

A chart can be simplified by chart moves.

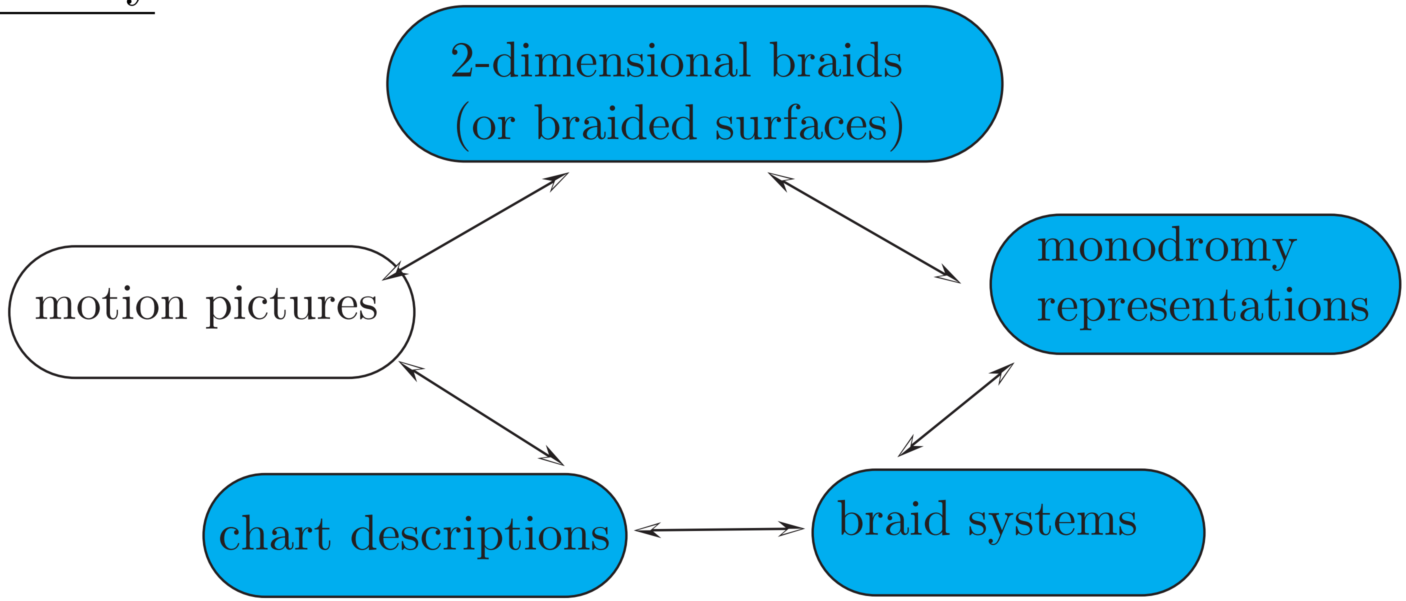
A (complicated) braid system $\mapsto$ a (complicated) chart description

$\mapsto$ a (simplified) chart description $\mapsto$ a (simplified) braid system



It is clear the last chart has a braid system $(\sigma_1, \sigma_1^{-1}, \sigma_3, \sigma_3^{-1})$.

Summary



A 2-dim braid can be described by the braid monodromy representation, and by a braid system. In general it is not easy to treat braid systems up to Hurwitz equivalence. Chart description helps this.

A (complicated) braid system $\mapsto$ a (complicated) chart description $\mapsto$ a (simplified) chart description $\mapsto$ a (simplified) braid system

Remarks

(1) 2-dim version of the Alexander and Markov Theorem

$\{$oriented links in $R^3\} \overset{1:1}{\Longleftrightarrow} \{$braids$\}/\sim$

Similarly, $\{$oriented surface-links in $R^4\} \overset{1:1}{\Longleftrightarrow} \{$2-dim braids$\}/\sim$

(Viro, Kamada, Matumoto)

(2) Lefschetz fibration

[Moishezon '77, Matsumoto '96] Lefschetz fibrations are determined monodromy representation, with some exception.

[K.-Matsumoto-Matumoto-Waki '04]

A simple proof to classification theorem of genus 1 Lefschetz fibrations (Y. Matsumoto '86).

$\{$ genus-1 LF $\} \overset{1:1}{\Longleftrightarrow} \{$ genus-1 LF monodromies $\} \overset{1:1}{\Longleftrightarrow} \{$ charts $\}/\sim$

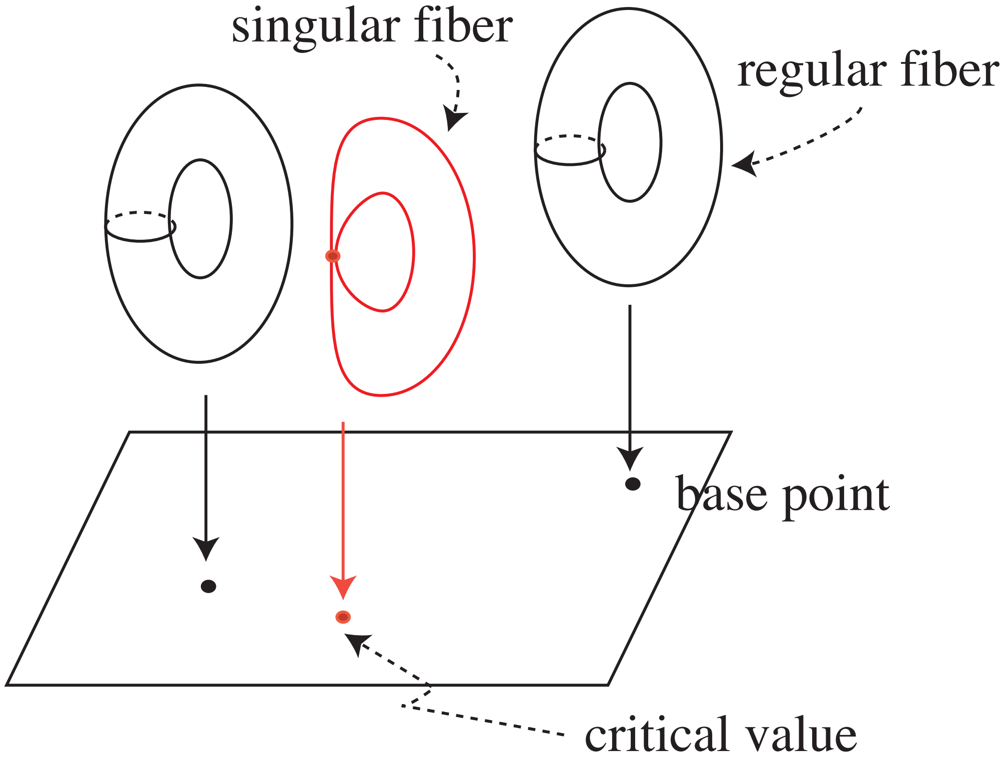
Lefschetz fibration

Let $f : M^4 \to S^2$ be a (smooth) genus g Lefschetz fibration:

- $\exists \{q_1, \ldots, q_r\} \subset S^2$ (critical values), $f| : M^4 \setminus f^{-1}\{q_1, \ldots, q_r\} \to S^2 \setminus \{q_1, \ldots, q_r\}$ is a smooth fibration of fiber genus g.
- Each singular fiber, $f^{-1}(q_i)$, contains one singular point, where f is $(z_1, z_2) \mapsto z_1^2 + z_2^2$.

A singular fiber is obtained by collapsing a simple loop (vanishing cycle) in a regular fiber. The local monodromy of a singular fiber is a positive or negative Dehn twist along the vanishing cycle.

We assume that f is 'relatively minimal' (i.e. all vanishing cycles are essential loops).



$$\{\text{Lefschetz fibrations}\}$$

$$\Longleftrightarrow \{\text{monodromy representations } \rho : \pi_1(S^2 \setminus \Delta) \to MC_g\}/\text{conj}$$

$$\Longleftrightarrow \{\text{Hurwitz systems, } (\rho(x_1), \ldots, \rho(x_r))\}/\text{conj and Hurwitz equiv.}$$

Basic Lefschetz fibrations of genus 2, f_0, f_1, f_2, f_I and f_{II}, are Lefschetz fibrations whose Hurwitz systems are

(1) $W_0 = (T)^2$ where $T = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1)$.

(2) $W_1 = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)^6$

(3) $W_2 = (\sigma), (\zeta_3, \zeta_4, \zeta_5, \zeta_2, \zeta_3, \zeta_4, \zeta_1, \zeta_2, \zeta_3)^2, (T)$

(4) $W_I = (\zeta_1, \zeta_1^{-1})$

(5) $W_{II} = (\sigma, \sigma^{-1})$.

LF	# of sing. fib.				chiral	irreducible
	n_{I}^+	n_{I}^-	n_{II}^+	n_{II}^-		
f_0	20	0	0	0	◯	◯
f_1	30	0	0	0	◯	◯
f_2	28	0	1	0	◯	×
f_I	1	1	0	0	×	◯
f_{II}	0	0	1	1	×	×

Theorem. Let f be a genus 2 Lefschetz fibration.

Suppose that $n^+_{\mathrm{II}}(f) \geq n^-_{\mathrm{II}}(f)$. Then

(1) $\mathcal{E}(f) := n^+_{\mathrm{I}}(f) - n^-_{\mathrm{I}}(f) - 28(n^+_{\mathrm{II}}(f) - n^-_{\mathrm{II}}(f))$ is a multiple of 10.

(We can define the parity, $\epsilon(f) \in \{0, 1\}$, by $\mathcal{E}(f)/10 \equiv \epsilon(f) \mod 2$.)

(2) $\exists m_0, \forall m \geq m_0, f\sharp(\overset{m}{\sharp} f_0) \cong (\overset{a+m}{\sharp} f_0)\sharp(\overset{b}{\sharp} f_1)\sharp(\overset{c}{\sharp} f_2)\sharp(\overset{d}{\sharp} f_I)\sharp(\overset{e}{\sharp} f_{II})$

(3) Moreover, $c = n^+_{\mathrm{II}}(f) - n^-_{\mathrm{II}}(f)$, $d = n^-_{\mathrm{I}}(f)$, $e = n^-_{\mathrm{II}}(f)$.

a, b are not determined uniquely. But we can take

$b = \epsilon(f) \in \{0, 1\}$, and then $a = (\mathcal{E}(f) - 30\epsilon(f))/20$.

Remark. We may take m_0 to be

$$n^-_{\mathrm{I}}(f) + 5\lfloor n^+_{\mathrm{II}}(f)/2 \rfloor + 1,$$

where

$$\lfloor k/2 \rfloor = \begin{cases} (m+1)/2 & (m : \text{odd}) \\ m/2 & (m : \text{even}) \end{cases}$$

If f is chiral and irreducible, then $n_{\mathrm{I}}^{-}(f) = n_{\mathrm{II}}^{+}(f) = n_{\mathrm{II}}^{-}(f) = 0$, and $m_0 = 1$. Thus, taking $m = 1$ in Thm, we have $f \sharp f_0 \cong (\overset{a+1}{\sharp} f_0) \sharp (\overset{b}{\sharp} f_1)$.

(B. Siebert and G. Tian (1999))

If f is chiral, then $n_{\mathrm{I}}^{-}(f) = n_{\mathrm{II}}^{-}(f) = 0$. By Thm, we have $f \sharp (\overset{m}{\sharp} f_0) \cong (\overset{a+m}{\sharp} f_0) \sharp (\overset{b}{\sharp} f_1) \sharp (\overset{c}{\sharp} f_2)$.

(D. Auroux (2003))

Our theorem is proved by use of a graphical method (chart description) of monodromy representations.

Summary A Lefschetz fibration can be described by the monodromy representation, by a braid system (Hurwitz system), or by a chart. We can use this for studies of LF's. For the case of genus 1 and 2, some known results can be easily proved by charts.

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