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Magnus representations of the mapping class group and L^2-torsion invariants

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Joint work with

- M. TAKASAWA-T. Morifuji, 2003–2004.
- T. Morifuji, 2006– in progress.

The main subjects of my talk;

- L^2-torsion,
  - Fuglede-Kadison determinant,
- Magnus representation.

1 Determinant in Linear Algebra

Recall one of the definitions of the determinant. Not standard, but well known in the are of zeta function theory, dynamical systems, or spectral geometry.

Fundamental formula:

$$\log|\det(B)|" = "\mathrm{tr}(\log(B)).$$

We want to explain more precisely the above.

Most simple case: A diagonal matrix.

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & & \dots & & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

Here we assume that the eigenvalues are

$$0 < \lambda_1, \dots, \lambda_n < 1.$$

Directly we compute,

$$\begin{aligned}\log(\det(B)) &= \log(\lambda_1 \cdots \lambda_n) \\ &= \sum_{i=1}^{n} \log(\lambda_i) \\ &= \sum_{i=1}^{n} \log(1 + (\lambda_i - 1)).\end{aligned}$$

Here recall the expansion of $\log(1+x)$ at $x = 0$

$$\log(1+x) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} x^p.$$

Then

$$\begin{aligned}
\log(\det(B)) &= \sum_{i=1}^{n}\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p}(\lambda_i - 1)^p\right)\\
&= -\sum_{i=1}^{n}\left(\sum_{p=1}^{\infty}\frac{1}{p}(1-\lambda_i)^p\right)\\
&= -\sum_{p=1}^{\infty}\frac{1}{p}\left(\sum_{i=1}^{n}(1-\lambda_i)^p\right)\\
&= -\sum_{p=1}^{\infty}\frac{1}{p}\operatorname{tr}\left((E-B)^p\right).
\end{aligned}$$

Hence, we can get the following equality:

$$\log\det(B) = -\sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}\left((E-B)^p\right),$$

or equivalently,

$$\det(B) = \exp\left(-\sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}\left((E-B)^p\right)\right).$$

General case:

- Non diagonal matrix case:
  - If B is a Hermite matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ s.t. $0 < \lambda_i < 1$, then the same formula holds.
  - If B is not a Hermite matrix, then we replace B to BB^*(B^*:the adjoint matrix of B).
    - For BB^*, each eigenvalue is changed from λ_i of B to $\lambda_i \bar{\lambda_i} = |\lambda_i|^2$ of BB^*.

In this case,

$$|\det(B)| = \exp\left(-\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{tr}(E - BB^*)^p\right).$$

– If some engenvalue $|\lambda_i| > 1$:
The problem is that the convergence radius of $\log(1+x)$ equals 1.

* For a sufficiently large constant $K > 0$ s.t. $0 < \lambda/K < 1$, then

$$\log(\lambda) = \log\left(K\frac{\lambda}{K}\right) = \log(K) + \log\left(\frac{\lambda}{K}\right).$$

* replace B to $\frac{1}{K}B$ (equivalently, BB^* to $\frac{1}{K^2}BB^*$).

Summary: For any matrix $B \in GL(n;\mathbb{C})$,

$$|\det(B)| = (K^2)^n \exp\left(-\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{tr}(E - \frac{1}{K^2}BB^*)^p\right).$$

This formula is one starting point to define the Fuglede-Kadison determinant.

2 Fuglede-Kadison determinant

We extend this equality to the one in the non commutative group algebra as the definition of $|\det(B)|$.

Origin: Theory of the von Neumann algebra.

- Fuglede-Kadison:Determinant theory in finite factor, Ann. of Math. (2), **55** (1952).

In this talk, we treat only group (von Neumann) algebra cases.

Here we fix some notations:

- π: a group.
- e: the unit of π.
- $\mathbb{Z}\pi$: the group algebra of π over $\mathbb{Z}$.
- $\mathbb{C}\pi$: the group algebra of π over $\mathbb{C}$(a linear space over $\mathbb{C}$).
- $l^2(\pi)$: l^2-completion of $\mathbb{C}\pi$, namely, algebra of all infinite sums $\sum_{g\in\pi} \lambda_g g$ s.t. $\sum_{g\in\pi} |\lambda_g|^2 < \infty$.

By using the equality $\log|\det| = \operatorname{tr}\log$, if we can define tr, we can do $\det$.
First the trace over $\mathbb{C}\pi$ is defined as follows.

Definition 2.1 $\mathbb{C}\pi$-trace:

$$\operatorname{tr}_{\mathbb{C}\pi}\left(\sum_{g\in\pi}\lambda_g g\right) = \lambda_e \in \mathbb{C}.$$

This $\mathbb{C}\pi$-trace $\operatorname{tr}_{\mathbb{C}\pi} : \mathbb{C}\pi \to \mathbb{C}$ can be naturally extended to the trace on the matrices over $\mathbb{C}\pi$.

For a matrix $B = (b_{ij}) \in M(n; \mathbb{C}\pi)$,

$$\mathrm{tr}_{\mathbb{C}\pi}(B) = \sum_{i=1}^{n} \mathrm{tr}_{\mathbb{C}\pi}(b_{ii}).$$

By using this trace

$$\mathrm{tr}_{\mathbb{C}\pi} : M(n; \mathbb{C}\pi) \to \mathbb{C},$$

Fuglede-Kadison determinant is defined as follows.

Definition 2.2 Fuglede-Kadison determinant:

$$\det_{\mathbb{C}\pi}(B) = K^{2n} \exp\left(-\frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p} \mathrm{tr}_{\mathbb{C}\pi} \left(E - \frac{BB^*}{K^2} \right)^p \right) \in \mathbb{R}_{>0}.$$

Here

- $K > 0$: a sufficiently large constant.
- $B^* = (\overline{b_{ji}})$: the adjoint matrix of $B = (b_{ij})$.

The adjoint matrix B^* is defined by the composite of two operations;

- the complex conjugate of coefficients,
- antihomomorphism :

$$\overline{\sum \lambda_g g} := \sum \bar{\lambda}_g g^{-1}.$$

Remark 2.3 The convergence of the infinite series is not trivial! However, the following fact is known.

If the above series for B converges, then its L^2-betti number of B:

$$\lim_{p \to \infty} \left(\frac{1}{p} \mathrm{tr}_{\mathbb{C}\pi} \left(\left(E - K^{-2} B B^* \right)^p \right) \right) = 0.$$

In many cases of groups, for example, a free group of a finite rank, an amenable group, or a hyperbolic group, if L^2-betti number is zero, then it converges.

3 Magnus representation

- $\Sigma_{g,1}$: an oriented compact surface of a genus $g \geq 1$ with 1 boundary component.
- $* \in \partial\Sigma_{g,1}$: a base point of $\Sigma_{g,1}$.
- $\mathcal{M}_{g,1} = \pi_0(\mathrm{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1}))$: the mapping class group of $\Sigma_{g,1}$.
- $\Gamma = \pi_1(\Sigma_{g,1}, *)$: free group of rank $2g$.
- $\langle x_1, \ldots, x_{2g} \rangle$: a generating system of Γ.

Proposition 3.1 (Dehn-Nielsen-Zieschang)

$$\mathcal{M}_{g,1} \ni \varphi \mapsto \varphi_* \in Aut(\Gamma)$$

is injective.

The Magnus representation of the mapping class group is defined as follows.

Definition 3.2 Magnus representation:

$$r : \mathcal{M}_{g,1} \ni \varphi \mapsto \left(\overline{\frac{\partial \varphi_*(x_j)}{\partial x_i}} \right)_{i,j} \in GL(2g; \mathbb{Z}\Gamma).$$

Here

- $\partial/\partial x_1, \ldots, \partial/\partial x_{2g} : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ are the Fox's free differentials.
- The conjugation on $\mathbb{Z}\Gamma$ is defined as follows. For any element $\sum_g \lambda_g g \in \mathbb{Z}\Gamma$,

$$\overline{\sum_g \lambda_g g} = \sum_g \lambda_g g^{-1}.$$

Remark 3.3 This map is not a homomorphism, but a crossed homomorphism. According to the practice, it is called the Magnus representation of $\mathcal{M}_{g,1}$.

By applying the Fuglede-Kadison determinant, we define the characteristic polynomial for the image of the Magnus representation.

4 L^2-torsion invariants

The problem is; How can we consider the variable t in the characteristic polynomial ?
The answer is to consider mapping torus and the variable "t" of the characteristic polynomial as the S^1-direction element in its fundamental group.
For the mapping class $\varphi \in \mathcal{M}_{g,1}$, we take its mapping torus

$$W_\varphi := \Sigma_{g,1} \times [0,1]/(x,1) \sim (\varphi(x),0).$$

From here, we put

$$\pi = \pi_1(W_\varphi, *).$$

We fix a base point

$$* \in \partial\Sigma_{g,1} \times \{0\} \subset \Sigma_{g,1} \times \{0\} \subset W_\varphi.$$

Now the group π has the following presentation:

$$\pi = \left\langle x_1, \cdots, x_{2g}, t \mid r_1, \ldots, r_n \right\rangle,$$

where $r_i := t x_i t^{-1} \left(\varphi_*(x_i)\right)^{-1} (i = 1, ..., 2g)$ and t is the generator of $\pi_1 S^1 \cong \mathbb{Z}$.

Here we can consider the characteristic polynomial

$$\det_{\mathbb{C}\pi}(tE - r(\varphi)) \in \mathbb{R}_{>0}$$

in the sense of Fuglede-Kadison determinant for any image $r(\varphi) \in M(2g; \mathbb{Z}\Gamma)$ of the Magnus representation.

By the theorem of Lück, it can be seen that it is the L^2-torsion of the 3-manifold W_φ for the regular representation of π.

- L^2-torsion [Lott, Lück, Carey, Mathai, ...] is a generalization of Reidemeister-Ray-Singer torsion to the torsion invariant with an infinite dimensional unitary representation.

Let us denote $\rho(\varphi)$ by the L^2-torsion of W_φ.

More precisely, we explain the Lück's formula. Applying the Fox free differentials to the relators $r_1, \cdots, r_{2g}$ of π, we obtain Fox matrix

$$A(\varphi) := \left(\frac{\partial r_i}{\partial x_j}\right)_{i,j} \in M(2g; \mathbb{Z}\pi).$$

By the theorem of Lück,

$$\begin{aligned}\log \rho(\varphi) &= -2 \log \det_{\mathbb{C}\pi}(A(\varphi)) \\ &= -2 \log \det_{\mathbb{C}\pi}(tE - \overline{{}^t r(\varphi)}) \\ &= -2 \log \det_{\mathbb{C}\pi}(tE - r(\varphi)).\end{aligned}$$

Then we can say that the characteristic polynomial of $r(\varphi)$ is the L^2-torsion of W_φ!

Here we want to ask the geometric meaning of L^2-torsion :

Answer:it is the hyperbolic volume!!

Theorem 4.1 (Lott, Schick,...) For any hyperbolic 3-manifold M,

$$\log \rho(M) = -\frac{1}{3\pi} vol(M).$$

Remark 4.2 Theoretically Lück's formula gives the way to compute the volume of the mapping torus W_φ for a given φ.

5 Series of L^2-torsion invariants

It is not easy to comput L^2-torsion. We want to find more computable invariant.

Fundamental framework:

Lower central series and nilpotent quotients!

Lower central series of Γ:

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots$$

- $\Gamma_1 = \Gamma$,
- $\Gamma_2 := [\Gamma_1, \Gamma_1]$,
- $\Gamma_k := [\Gamma_{k-1}, \Gamma_1]$ $(k \geq 2)$.
- $N_k := \Gamma/\Gamma_k$:k-th nilpotent quotient
- $p_k : \Gamma \to N_k$: natural projection.

Under these situations, we obtain the series of (graded) Magnus representations:

$$r_k : \mathcal{M}_{g,1} \to GL(2g; \mathbb{Z}N_k).$$

In the case of $k = 1$, the projection is

$$p_1 : \Gamma = \pi_1(\Sigma_{g,1}) \to N_1 = \{e\},$$

then we obtain the map

$$r_1 : \mathcal{M}_{g,1} \to GL(2g; \mathbb{Z}),$$

which is just the homology representation of the mapping class group.

In the case of $k = 2$, the projection is

$$\Gamma = \pi_1(\Sigma_{g,1}) \to H = H_1(\Sigma_{g,1}; \mathbb{Z}),$$

then we obtain the map

$$r_2 : \mathcal{M}_{g,1} \to GL(2g; \mathbb{Z}H).$$

If we restrict this map to the Torelli group

$$\mathcal{I}_{g,1} = \mathrm{Ker}\{\mathcal{M}_{g,1} \to \mathrm{Sp}(2g; \mathbb{Z})\},$$

then

$$r_2 : \mathcal{I}_{g,1} \to GL(2g; \mathbb{Z}H)$$

is a homomorphism.

Because $\Gamma_k \lhd \Gamma$, then

$$\pi(k) = \pi/\Gamma_k \cong N_k \rtimes \mathbb{Z}.$$

Then

$$p_k : \pi \longrightarrow \pi(k) = N_k \rtimes \mathbb{Z},$$

and we write $p_{k*} : \mathbb{C}\pi \longrightarrow \mathbb{C}\pi(k)$ to its induced homomorphism on the group ring. The k-th Fox matrix

$$A_k := \left(p_{k*} \left(\frac{\partial r_i}{\partial x_j} \right) \right) \in M(2g; \mathbb{C}\pi(k)).$$

[**Morifuji-Takasawa-Kitano**].
For W_φ, k-th L^2-torsion invariant $\rho_k(\varphi)$ can be defined and **,**

$$\log \rho_k(\varphi) = -2 \log \mathrm{det}_{\mathbb{C}\pi(k)}(A_k).$$

Remark 5.1 The k-th invariant $\rho_k(\varphi)$ can be counted the characteristic polynomial of a matrix $r_k(\varphi) \in GL(2g; \mathbb{C}\pi(k))$ in terms of Fuglede-Kadison determinant.

The first and fundamental problem is the following.

Problem 5.2

$$\lim_{k \to \infty} (\rho_k(\varphi)) = \rho(\varphi)?$$

It is still the open problem. We explain some computation and some partial results about these invariants.

6 Example

We compute the first and second invariant for the torefoile knot exterior. Before doing this, we can see the following by the general theory of torsion invariants.

Proposition 6.1 $\log \rho_k(\varphi^n) = n \log \rho_k(\varphi)$.

Remark 6.2 Topologically, $W_{\varphi^n} \to W_\varphi$ is an n-fold cyclic covering of W_φ.

Because L^2-torsion is a topological invariant of mapping torus, then we can see the following.

Proposition 6.3 For any mapping class $\varphi \in \mathrm{Ker}\{\mathcal{M}_{g,1} \to \mathcal{M}_g\}$, it holds that

$$\log \rho_k(\varphi) = 0.$$

Remark 6.4 For φ as above, topologically $W_\varphi \cong \Sigma_{g,1} \times S^1$.

Now we consider the torefoile knot exterior $E(3_1)$.
First $E(3_1)$ admits a strcture of $\Sigma_{1,1}$-bundle over S^1. Then there exists $\varphi \in \mathcal{M}_{1,1}$ s.t.

$$E(3_1) = W_\varphi.$$

We can choice φ s.t. φ^6 is the Dehn twist along the boudary parallel simple closed curve, that is, $\varphi^6 \in \mathrm{Ker}(\mathcal{M}_{1,1} \to \mathcal{M}_1)$.

On $\pi_1(\Sigma_{1,1}) = \langle x_1, x_2 \rangle$, we can do that

$$\varphi_*^6(x_1) = [x_1, x_2] x_1 [x_1, x_2]^{-1},$$
$$\varphi_*^6(x_2) = [x_1, x_2] x_2 [x_1, x_2]^{-1}.$$

Then

$$\pi_1(W_{\varphi^6}) = \langle x_1, x_2, t |\ t x_1 t^{-1} = \omega x_1 \omega^{-1}, \\ t x_2 t^{-2} = \omega x_2 \omega^{-1} \rangle$$

where $\omega = [x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}$.

For this case, we can directly compute the Fox free differentials

$$\frac{\partial}{\partial x_1}(tx_1t^{-1} - \omega x_1\omega^{-1}),$$

$$\frac{\partial}{\partial x_2}(tx_2t^{-1} - \omega x_2\omega^{-1}),$$

and obtain the Fox matrix $A(\varphi^6)$ as follows.

$$A(\varphi^6) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2; \mathbb{Z}\pi_1(W_{\varphi^6}))$$

where

$$a_{11} = t - 1 + xyx^{-1} - \omega + \omega x \omega^{-1}(1 - xyx^{-1})$$

$$a_{12} = -x + \omega + \omega x \omega^{-1}(x - \omega)$$

$$a_{21} = -1 + xyx^{-1} + \omega y \omega^{-1}(1 - xyx^{-1})$$

$$a_{22} = t - x + \omega y \omega^{-1}(x - \omega).$$

- Computation of ρ:
  It is hard to compute the Fuglede-Kadison determinant by definition. However we know that $W_\varphi = E(3_1)$ and W_{φ^6} do not admit hyperbolic structures. Then

$$\log \rho(\varphi) = \log \rho(\varphi^6) = 0.$$

- $k = 1$ case:
  In this case, by applying the projection $\pi = \pi_1(W_{\varphi^6}) \to \pi(1) = \langle t \rangle$, we can obtain

$$A_1(\varphi^6) = \begin{pmatrix} t-1 & 0 \\ 0 & t-1 \end{pmatrix}.$$

Then

$$\begin{aligned}\log \rho_1(\varphi^6) &= -2 \log \det_{\mathbb{C}\pi(1)}(A_1(\varphi^6)) \\ &= -2 \log \det_{\mathbb{C}\pi(1)}(tE - E) \\ &= 0 \text{ by computation.}\end{aligned}$$

On the other hand, we have seen that

$$\log \rho_1(\varphi^6) = 6 \log \rho_1(\varphi) = 0.$$

Therefore we can see

$$\log \rho_1(\varphi) = \log \rho_1(E(3_1)) = 0.$$

- $k = 2$ case:
  In this case, we can reduce $A(\varphi^6)$ to

$$A_2(\varphi^6) = \begin{pmatrix} t + x + y - xy - 2 & x^2 - 2x + 1 \\ -y^2 + 2y - 1 & t + xy - x - y \end{pmatrix}.$$

Then

$$\begin{aligned} \log \rho_1(\varphi^6) &= -2 \log \det_{\mathbb{C}\pi(2)}(A_2) \\ &= 0 \text{ by computation.} \end{aligned}$$

On the other hand,

$$\log \rho_2(\varphi^6) = 6 \log \rho_2(\varphi) = 0.$$

Then

$$\log \rho_2(\varphi) = \log \rho_2(E(3_1)) = 0.$$

In the genus 1 case, the following holds for any mapping class.

Theorem 6.5 (MTK)

$$\log \rho_1(\varphi) = -2 \log \max\{1, |\alpha|, |\alpha|^{-1}\}.$$

Here

α, α^{-1} : the eigenvalues of $\varphi_* \in SL(2; \mathbb{Z})$.

Here we consider the following family of matricies

$$\varphi_* = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$$

where $\mathrm{tr}(\varphi_*) = a$.
We compute ρ_1 and values of hyperbolic volume (by using SnapPea) and compare them.

$\mathrm{tr}(\mathrm{r}_1(\varphi)) = \mathrm{a}$	$-3\pi \log \rho_1(\varphi)$	volume
0	0	0
1	0	0
2	0	0
3	18.1412	2.0298
4	24.8240	2.6667
5	29.5334	2.9891
6	33.2270	3.1772
7	36.2825	3.2969
8	38.8948	3.3775

In the case of $k = 2$, we can prove the following.

Theorem 6.6 (MTK) For any $\varphi \in \mathcal{M}_{1,1}$,

$$\log \rho_2(\varphi) = 0.$$

Furthermore we can prove the following.

Theorem 6.7 (MK) If $\varphi \in \mathcal{M}_{1,1}$ is a Dehn twist which is a twist along a non-separating curve(=not parallel to the boundary), then

$$\log \rho_k(\varphi) = 0.$$

We remark that $vol(W_\varphi) = 0$ for such φ. Therefore it is trivial in some sense, but it holds that

$$\log \rho_k(\varphi) = \log \rho(\varphi)$$

for such φ.

Remark 6.8 We have not yet computed the third invariant ρ_3 for non trivial examples. It is very hard to compute!

7 Results

Recall Nielsen-Thurston classification of the mapping classes:

- periodic
- reducible
- pseudo Anosov

In $\mathcal{M}_{g,1}$, there is no periodic element. However, for a lift of a periodic element in $\mathcal{M}_g$,
i.e., for any mapping class $\varphi \in \mathcal{M}_{g,1}$ such that

$\varphi^n \in$ Ker$\{\mathcal{M}_{g,1} \to \mathcal{M}_g\}$, we can see the following.

Proposition 7.1 For any φ s.t. $\varphi^n \in$ Ker, then $\log \rho_k(\varphi) = 0$.

Proposition 7.2 For $\varphi \in \mathcal{M}_{g,1}$, if there exists a separating simple closed curve $\gamma \subset \Sigma_{g,1}$ such that φ fixes pointwisely γ**,** then $\rho_k(\varphi)$ can be computed by the both of L^2-torsion invariants of 2 compact surfaces cutted by γ.

Remark 7.3 In the above case, we can reduce computation to the one for lower genus cases.

$k = 1$ case,

- $\pi(1) = \pi/\Gamma_1 = N_1 \rtimes \mathbb{Z} \cong \mathbb{Z}$
- $GL(2g; \mathbb{Z}\pi(1)) = GL(2g; \mathbb{Z})$.

Here Magnus representation is just

$$r_1 : \mathcal{M}_{g,1} \to Sp(2g : \mathbb{Z}).$$

This Fuglede-Kadison determinant is the usual determinant over $\mathbb{C}[\mathbb{Z}]$. ρ_1 can be computed as follows.

Theorem 7.4 (Lott, Lück, MTK) For any $\varphi \in \mathcal{M}_{g,1}$,

$$\log \rho_1(\varphi) = \int_{S^1} \log |\det(tE - r_1(\varphi))| dt.$$

Remark 7.5 This integration is the Mahler measure for 1-variable polynomials.

By properties of Mahler measure, we can see the following.

Corollary 7.6

$$\log \rho_1(\varphi) = -2 \sum_{i=1}^{2g} \log \max\{1, |\alpha_i|\}$$

Here

$\alpha_1, \ldots, \alpha_{2g}$: the eigenvalues of $r_1(\varphi) \in Sp(2g;\mathbb{Z})$.

In the genus 1 case, 2 eigenvalues of $r_1(\varphi) \in SL(2;\mathbb{Z})$ can be determined by the trace. Now we see the following corollary.

Corollary 7.7 It holds;

$$\log \rho_1(\varphi) = 0 \Leftrightarrow |\mathrm{tr}(r_1(\varphi))| < 2.$$

Recall that $-3\pi \log \rho(\varphi) = vol(W_\varphi)$, then now we compare the volume with $-3\pi \log \rho_1(\varphi)$.

Problem 7.8 For A such that $|\mathrm{tr}(A)| > 2$, compute ρ_k and investigate its behavior when $k \to \infty$.

Higher genus case:

$k = 2$, Magnus representation

$$r_2 : \mathcal{I}_{g,1} \to GL(2g; \mathbb{Z}H).$$

The mapping class φ_* acts on H trivially,

$$\pi(2) = H \times \mathbb{Z},$$

then $\mathbb{C}\pi(2)$ is a commutative ring. Hence, we can use the usual determinant.

Under this situation,

Theorem 7.9 (MTK) $\log \rho_2$ can be described by using the Mahler measure for multi-variable polynomials.

From computation by M. Suzuki,

Corollary 7.10 If φ is a BP-map, or a BSCC-map, then

$$\log \rho_2(\varphi) = 0.$$

Remark 7.11 There exists $\varphi \in \mathcal{I}_{g,1}$ such that

$$\log \rho_2(\varphi) \neq 0.$$

Theorem 7.12 (MK) If $\varphi \in \mathcal{M}_{g,1}$ is a product of Dehn twists along any disjoint non-separating simple closed curves which are mutually non-homologous, then

$$\log \rho_k(\varphi) = 0.$$