Conjugacy Classes of Symmetric Braids

Sang-Jin Lee

Konkuk University

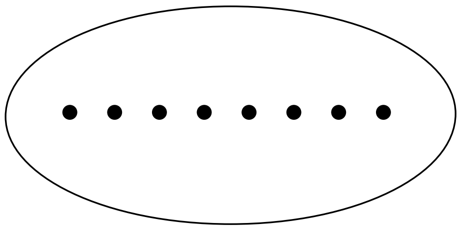
(Jointwork with Eon-Kyung Lee)

Reference: E.-K. Lee and S.-J. Lee, *Injectivity on the set of conjugacy classes of some monomorphisms between Artin groups*, arXiv:0802.2314.

Braids in Paris
Sep. 17 – 20, 2008

Braid groups

Let D_n be the n-punctured disk $\{z \in \mathbb{C} : |z| \leq n+1\} \setminus \{1, 2, \ldots, n\}$.

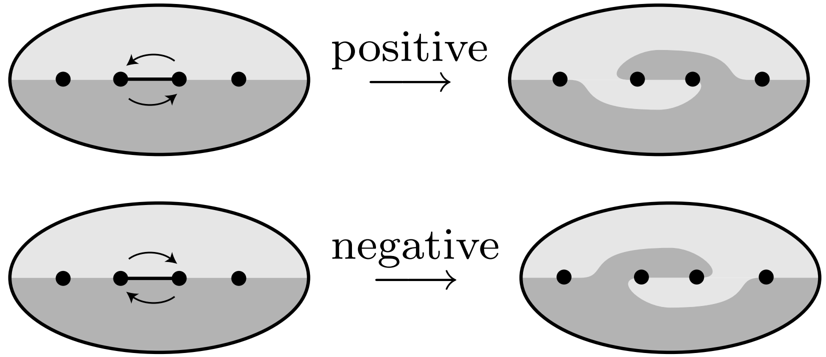


The **n-braid group** B_n is the group of self-diffeomorphisms of D_n that fix ∂D^2 pointwise, modulo isotopy relative to ∂D^2.

Let σ_i denote the positive Dehn-twist along the arc $\overline{i(i+1)}$.

σ_2:

σ_2^{-1}:

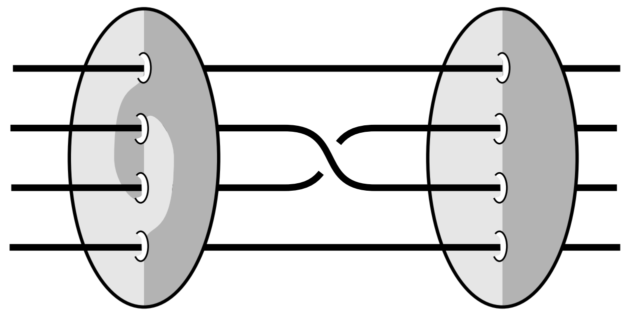


Then B_n has the following presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \end{array} \right\rangle.$$

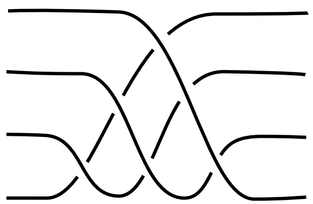
An n-braid can be considered as a collection of n strings $l = l_1 \cup \cdots \cup l_n$ in $[0,1] \times D^2$ such that

$$|l \cap (t \times D^2)| = n \text{ for all } t \in [0,1];$$
$$l \cap (t \times D^2) = t \times \{1, \ldots, n\} \text{ for } t = 0, 1.$$



Periodic braids

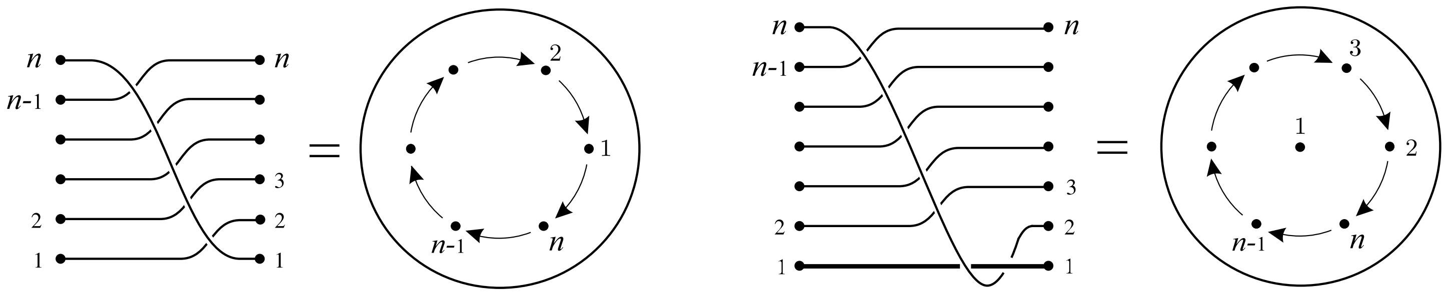
Let Δ be the half-twist $\sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{n-1}\cdots\sigma_1)$.
The center of B_n is the infinite cyclic group $\langle\Delta^2\rangle$ generated by Δ^2.



$$\Delta = \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1) \in B_4$$

An n-braid ω is said to be **periodic** if it has a central power.

Let $\delta = \delta_{(n)} = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$ and $\epsilon = \epsilon_{(n)} = (\sigma_{n-1}\cdots\sigma_1)\sigma_1$. Then $\delta^n = \epsilon^{n-1} = \Delta^2$, hence δ and ϵ are periodic braids.



(a) $\delta = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$ (b) $\epsilon = (\sigma_{n-1}\cdots\sigma_1)\sigma_1$

Kerékjártó-Brower-Eilenberg's Theorem

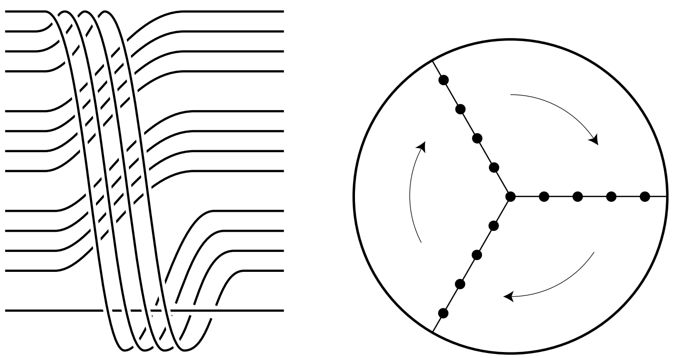
A braid is periodic iff it is conjugate to δ^k or ϵ^k for some $k \in \mathbb{Z}$.

Remarks.

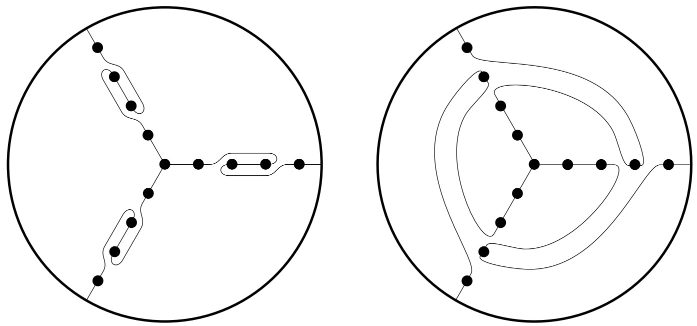
Let ω be a periodic braid. Then

ω is represented by $(\frac{2\pi}{m})$-rotation of a punctured disk for some $m \geq 1$;

the elements of $Z(\omega)$ are represented by equivariant diffeomorphisms.



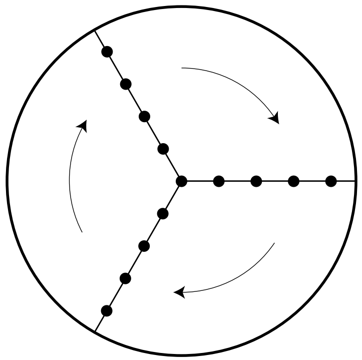
$\omega = \epsilon^4 \in B_{13}$



elements in $Z(\omega)$

Question.

Let ρ be the $(\frac{2\pi}{m})$-rotation of a punctured disk for some $m \geq 2$.
Let f and g be ρ-equivariant diffeomorphisms of the punctured disk.
If they are conjugate by a diffeomorphism of the punctured disk,
then are they conjugate by a ρ-equivariant diffeomorphism?



Equivalent question.

Let $\omega \in B_n$ be a periodic braid.
Let $\alpha, \beta \in Z(\omega)$ be conjugate in B_n.
Are they conjugate in $Z(\omega)$?

Main Theorem

Let $\omega \in B_n$ be a periodic braid. Let $\alpha, \beta \in Z(\omega)$.
If α and β are conjugate in B_n, then they are conjugate in $Z(\omega)$.

In other words, the inclusion $Z(\omega) \to B_n$ induces an injective function on the set of conjugacy classes.

Remark.
Let $\alpha, \beta \in Z(\omega)$ and $\beta = \gamma^{-1}\alpha\gamma$ for some $\gamma \in B_n$.
Then the conjugating element γ does not necessary belong to $Z(\omega)$.
The above theorem says that there is another n-braid $\gamma_1 \in Z(\omega)$ such that $\beta = \gamma_1^{-1}\alpha\gamma_1$.

Idea of proof

Nielsen-Thurston Classification Theorem.

Automorphisms of a surface with negative Euler characteristic are, up to isotopy, either

1. **pseudo-Anosov** ($\exists$ a transverse pair of invariant measured foliations);
2. **periodic** ($f^k = \mathrm{id}$ for some $k > 0$);
3. **reducible** ($\exists$ an essential curve system whose isotopy class is invariant under the automorphism).

Pseudo-Anosov case

Theorem (González-Meneses and Wiest 2004).

If $\alpha \in B_n$ is pseudo-Anosov, then $Z(\alpha)$ is free abelian of rank two.

Lemma. Let $\omega \in B_n$ be periodic.
If $\alpha \in Z(\omega)$ is pseudo-Anosov, then $Z(\alpha) \subset Z(\omega)$.

Proof. Let $\gamma \in Z(\alpha)$. Note that $\omega \in Z(\alpha)$.
Because $Z(\alpha)$ is abelian, γ commutes with ω. QED

Theorem (Main theorem for pseudo-Anosov case).

Let $\omega \in B_n$ be periodic, and let $\alpha, \beta \in Z(\omega)$ be pseudo-Anosov. Let $\beta = \gamma\alpha\gamma^{-1}$ for $\gamma \in B_n$. Then $\gamma \in Z(\omega)$.

This means that any conjugating braid from α to β belongs to $Z(\omega)$.

Proof. Let x^y denote $y^{-1}xy$.
Note that $\alpha^\omega = \alpha$ and $\beta^\omega = \beta$, and ω^m is central for some $m \geq 1$.

$$\begin{aligned} &\beta = \gamma\alpha\gamma^{-1} \text{ and } \beta = \gamma^\omega \alpha\, (\gamma^\omega)^{-1} \\ &\quad \Rightarrow (\gamma^{-1}\gamma^\omega)\alpha = \alpha(\gamma^{-1}\gamma^\omega) \\ &\quad \Rightarrow (\gamma^{-1}\gamma^\omega) \in Z(\alpha) \\ &\quad \Rightarrow (\gamma^{-1}\gamma^\omega) \in Z(\omega) \end{aligned}$$

$$\begin{aligned} (\gamma^{-1}\gamma^\omega)^m &= (\gamma^{-1}\gamma^\omega)\cdots(\gamma^{-1}\gamma^\omega) \\ &= (\gamma^{-1}\gamma^\omega)(\gamma^{-1}\gamma^\omega)^\omega \cdots (\gamma^{-1}\gamma^\omega)^{\omega^{m-1}} \\ &= \gamma^{-1}\gamma^\omega \cdot (\gamma^\omega)^{-1}(\gamma^{\omega^2})\cdots(\gamma^{\omega^{m-1}})^{-1}(\gamma^{\omega^m}) \\ &= \gamma^{-1}\gamma^{\omega^m} = \gamma^{-1}\gamma = 1. \end{aligned}$$

Because B_n is torsion-free, we have $\gamma^{-1}\gamma^\omega = 1$, hence $\gamma = \gamma^\omega$. QED

Periodic case

Theorem (González-Meneses 2003, Lee-Lee 2007)

Let G be one of the Artin groups of type $\mathbf{A}$, $\mathbf{B}$, $\tilde{\mathbf{A}}$, $\tilde{\mathbf{C}}$. Let $\alpha, \beta \in G$.
If $\alpha^k = \beta^k$ for some $k \neq 0$, then α and β are conjugate in G.

Theorem (Bessis-Digne-Michel 2002)

Let $\omega \in B_n$ be non-central and periodic.
Then $Z(\omega)$ is isomorphic to an Artin group of type $\mathbf{B}$.

Proof of Main Theorem for periodic case.

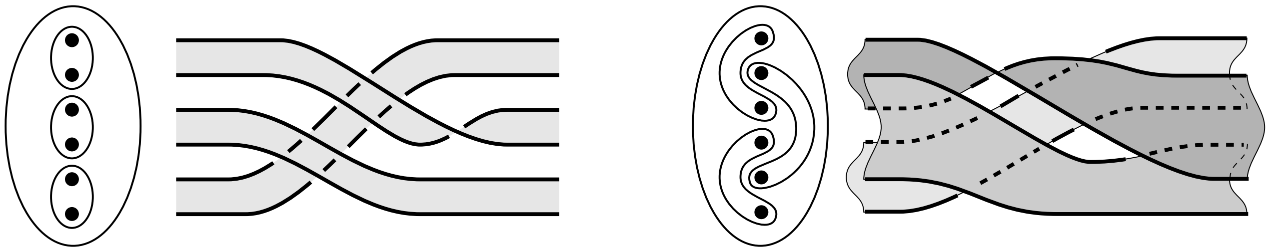
Let $\alpha, \beta \in Z(\omega)$ be periodic braids which are conjugate in B_n.

There is an integer $k \neq 0$ such that both α^k and β^k are central.
Then $\alpha^k = \beta^k$ because α^k and β^k are central and conjugate in B_n,

By the above theorems, α and β are conjugate in $Z(\omega)$. QED

Reducible case

$\alpha \in B_n$ is **reducible** if $\exists$ an essential curve system $\mathcal{C} \subset D_n$ such that $\alpha(\mathcal{C}) = \mathcal{C}$. ($\mathcal{C}$ is called a **reduction system** of α.)



(a) standard reduction system (b) non-standard reduction system

Remarks.

(Birman-Lubotzky-McCarthy '83) For each reducible mapping class, there is a unique **canonical reduction system**.

A curve system is **standard** if each component is isotopic to a round circle. If a reducible braid has a **standard** canonical reduction system, it is easy to decompose it into exterior and interior braids.

Let $\mathcal{R}(x)$ denote the the canonical reduction system of $x \in B_n$.

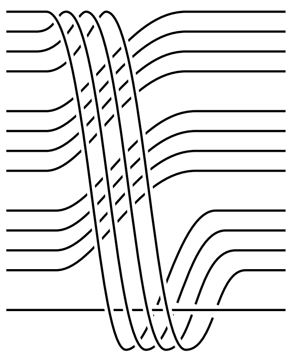
Idea of Proof of Main Theorem for reducible case.

Let (ω, α, β) be a triple such that ω is a periodic n-braid and $\alpha, \beta \in Z(\omega)$ are reducible braids conjugate in B_n.

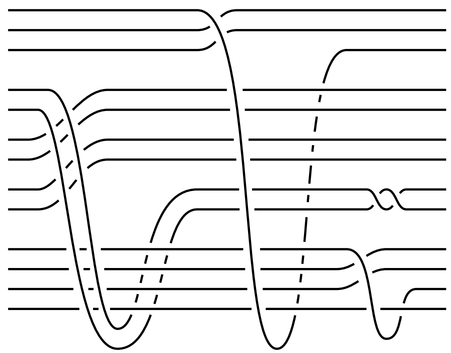
Step I.
Prove that there is a triple $(\omega', \alpha', \beta')$ such that

$\omega' = x\omega x^{-1}$ for some $x \in B_n$;
$\alpha' = xx_1\alpha x_1^{-1}x^{-1}$ and $\beta' = xx_2\beta x_2^{-1}x^{-1}$ for some $x_1, x_2 \in Z(\omega)$,
hence $\alpha', \beta' \in Z(\omega')$ and they are conjugate in B_n;
$\mathcal{R}(\alpha') = \mathcal{R}(\beta')$ and they are standard.

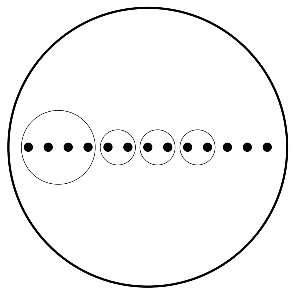
The periodic braid ω' obtained in Step I is a specific periodic braid depending only on the conjugacy class of α and β.



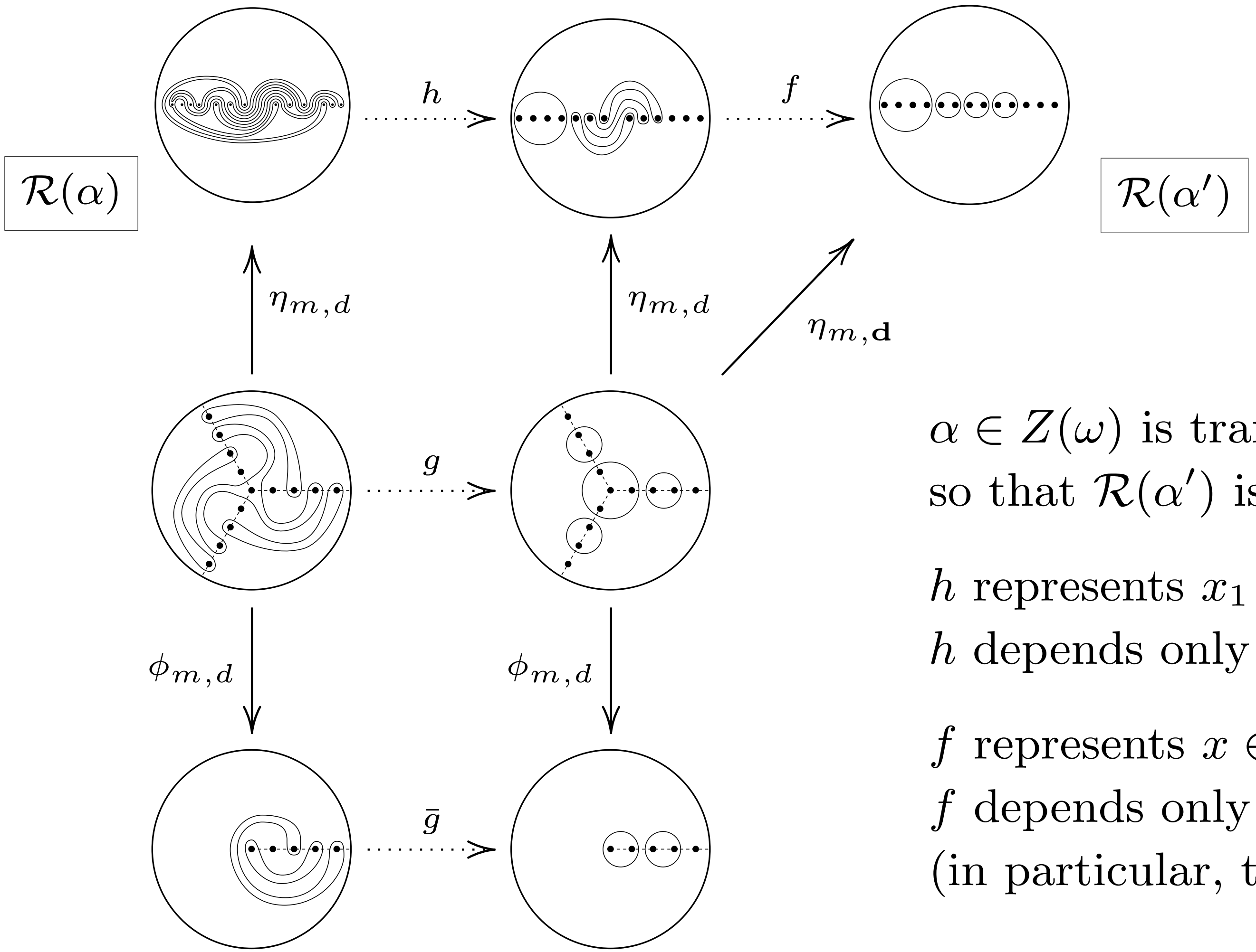
(a) $\omega = \epsilon^4 \in B_{13}$



(b) $\omega' = \mu_{3,(2,2,1)}$



(c) $\mathcal{R}(\alpha') = \mathcal{R}(\beta')$



$\alpha \in Z(\omega)$ is transformed to $\alpha' \in Z(\omega')$ so that $\mathcal{R}(\alpha')$ is standard.

h represents $x_1 \in Z(\omega)$;
h depends only on $\mathcal{R}(\alpha)$.

f represents $x \in B_n$;
f depends only on the type of $\mathcal{R}(\alpha)$
(in particular, the conjugacy class of α).

Step II.

By Step I, we may assume:

> ω is a specific periodic braid obtained in Step I;
> $\alpha, \beta \in Z(\omega)$ are reducible with $\mathcal{R}(\alpha) = \mathcal{R}(\beta)$ being standard.

By previous results, we may assume:

> Main theorem holds for pseudo-Anosov and periodic braids;
> α and β are pure braids.

Use induction on braid index.

Application to Artin groups

There are well-known monomorphisms between some Artin groups.

Question.
Do they induce injective functions on the set of conjugacy classes?

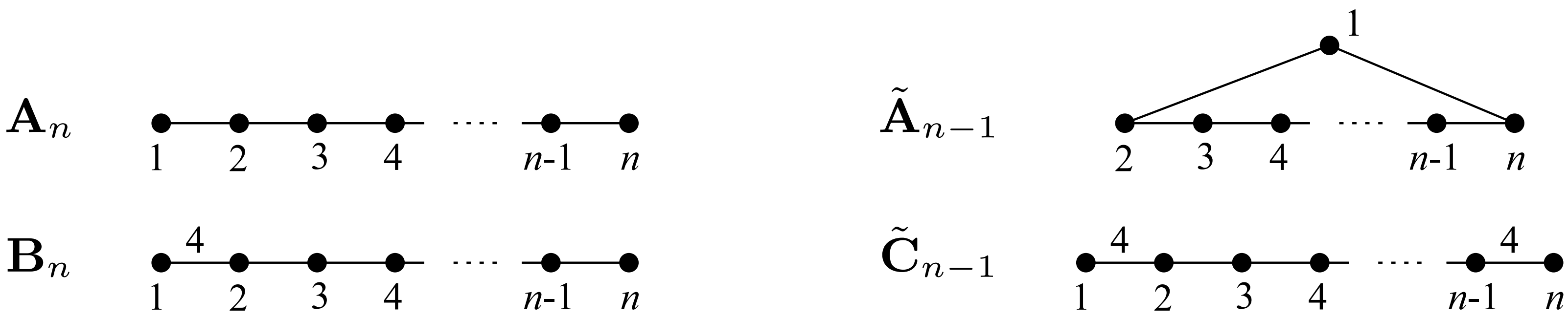
Let M be a symmetric $n \times n$ matrix with entries $m_{ij} \in \mathbb{N} \cup \{\infty\}$ where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$.

The **Artin group of type** M is defined by the presentation

$$A(M) = \langle s_1, \ldots, s_n \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \quad \text{for all } i \neq j,\, m_{ij} \neq \infty \rangle.$$

It is convenient to define an Artin group by a **Coxeter graph**:

vertices are numbered $1, \ldots, n$;
there is an edge labelled m_{ij} between the vertices i and j if $m_{ij} \geq 3$ or $m_{ij} = \infty$. (The label 3 is usually suppressed.)



Coxeter graphs of type $\mathbf{A}_n$, $\mathbf{B}_n$, $\tilde{\mathbf{A}}_{n-1}$ and $\tilde{\mathbf{C}}_{n-1}$.

For example, $A(\mathbf{A}_n)$ is isomorphic to the $(n+1)$-braid group B_{n+1}:

$$B_{n+1} = \left\langle \sigma_1, \ldots, \sigma_n \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| \geq 2, \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{if } |i-j| = 1. \end{array} \right\rangle.$$

Theorem (Bessis-Digne-Michel 2002)

Let $\omega \in B_n$ be non-central and periodic.
Then $Z(\omega)$ is isomorphic to $A(\mathbf{B}_d)$, the Artin group of type $\mathbf{B}_d$,
where $d = \begin{cases} \gcd(k,n) & \text{if } \omega \text{ is conjugate to } \delta^k, \\ \gcd(k,n-1) & \text{if } \omega \text{ is conjugate to } \epsilon^k. \end{cases}$

By the above theorem, there are monomorphisms
$A(\mathbf{B}_d) \to A(\mathbf{A}_{md})$ and $A(\mathbf{B}_d) \to A(\mathbf{A}_{md-1})$:

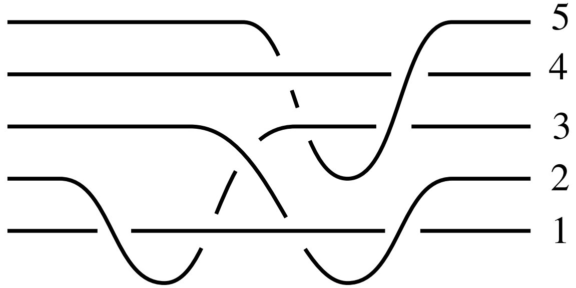
$$\begin{aligned} A(\mathbf{B}_d) &\simeq Z(\epsilon^d_{(md+1)}) \subset B_{md+1} = A(\mathbf{A}_{md}); \\ A(\mathbf{B}_d) &\simeq Z(\delta^d_{(md)}) \subset B_{md} = A(\mathbf{A}_{md-1}). \end{aligned}$$

Corollary of Main Theorem.

The monomorphisms $A(\mathbf{B}_d) \to A(\mathbf{A}_{md})$ and $A(\mathbf{B}_d) \to A(\mathbf{A}_{md-1})$ induce injective functions on the set of conjugacy classes.

For $P \subset \{1, \ldots, n\}$, we say that $\alpha \in B_n$ is **P-pure** if $\pi_\alpha(i) = i$ for each $i \in P$, where π_α denotes the induced permutation of α.

A braid α is said to be **1-unlinked** if it is 1-pure and the linking number of the first strand with the other strands is zero.



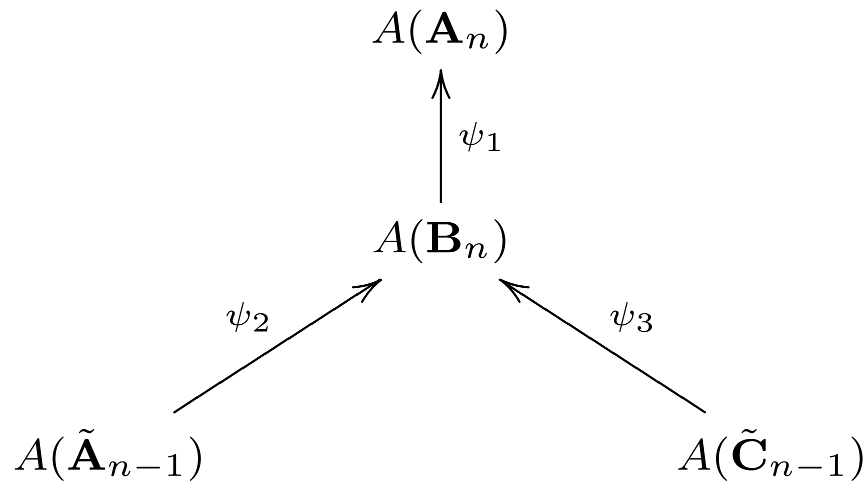
This braid is $\{1, 4, 5\}$-pure and 1-unlinked

Let $B_{n+1,1}$ denote the subgroup of B_{n+1} consisting of 1-pure braids.

Theorem (Allcock 2002, Charney-Crisp 2005, ...)

$$\begin{array}{rcllcl} A(\mathbf{B}_n) & \simeq & \{\alpha \in B_{n+1} \mid \alpha \text{ is 1-pure}\} = B_{n+1,1} & \subset & B_{n+1} \simeq A(\mathbf{A}_n); \\ A(\tilde{\mathbf{A}}_{n-1}) & \simeq & \{\alpha \in B_{n+1} \mid \alpha \text{ is 1-unlinked}\} & \subset & B_{n+1,1} \simeq A(\mathbf{B}_n); \\ A(\tilde{\mathbf{C}}_{n-1}) & \simeq & \{\alpha \in B_{n+1} \mid \alpha \text{ is } \{1, n+1\}\text{-pure}\} & \subset & B_{n+1,1} \simeq A(\mathbf{B}_n). \end{array}$$

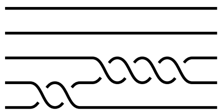
The above inclusions induce monomorphisms



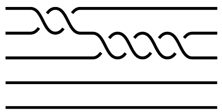
Theorem.

None of the monomorphisms $A(\mathbf{B}_n) \xrightarrow{\phi_1} A(\mathbf{A}_n)$, $A(\tilde{\mathbf{A}}_{n-1}) \xrightarrow{\phi_2} A(\mathbf{B}_n)$ and $A(\tilde{\mathbf{C}}_{n-1}) \xrightarrow{\phi_3} A(\mathbf{B}_n)$ is injective on the set of conjugacy classes.

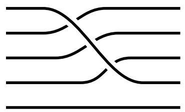
We prove the theorem by giving examples. $(n = 4)$



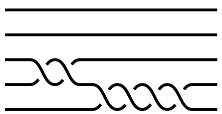
(a) $\alpha_1 = \sigma_1^2\sigma_2^4$



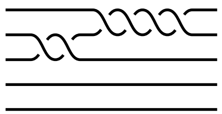
(b) $\alpha_2 = \sigma_n^2\sigma_{n-1}^4$



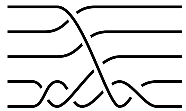
(c) $\alpha_3 = \sigma_n \cdots \sigma_2$



(d) $\beta_1 = \sigma_2^2\sigma_1^4$



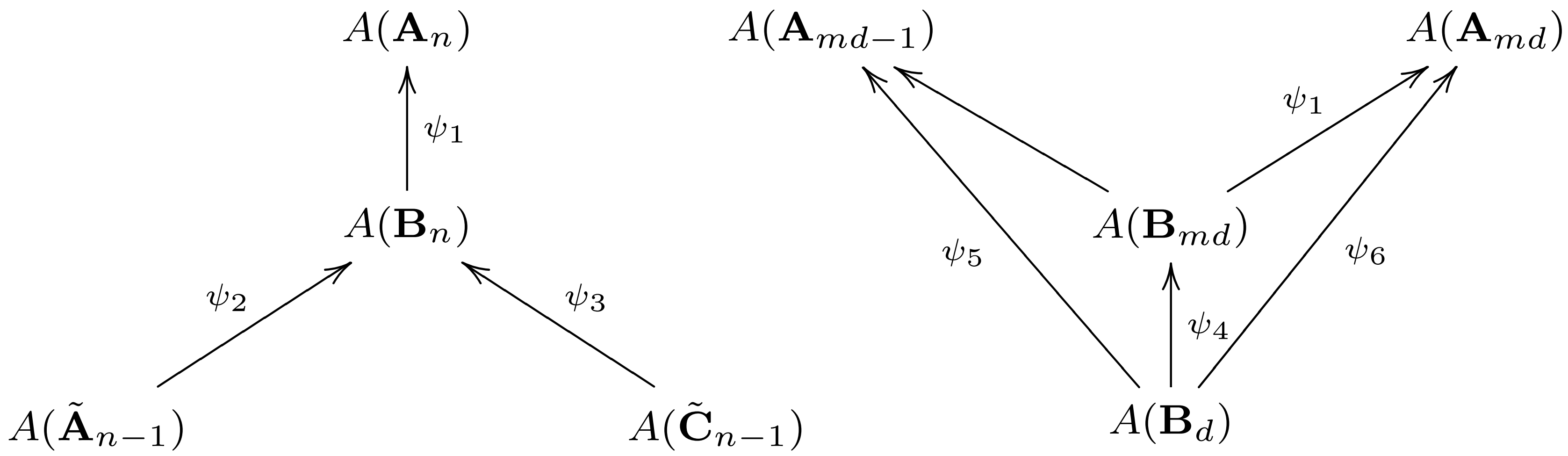
(e) $\beta_2 = \sigma_{n-1}^2\sigma_n^4$



(f) $\beta_3 = \sigma_1^{-2}\alpha_3\sigma_1^2$

(α_1, β_1): 1-pure, conjugate in B_5, not conjugate by a 1-pure braid.
(α_2, β_2): $\{1, 5\}$-pure, conjugate in $B_{5,1}$, not conjugate by a $\{1, 5\}$-pure braid.
(α_3, β_3): 1-unlinked, conjugate in $B_{5,1}$, not conjugate by a 1-unlinked braid.

Summary on the injectivity on the set of conjugacy classes



$\psi_1, \ldots, \psi_6$ are monomorphisms between Artin groups.

ψ_4, ψ_5, ψ_6 are injective on the set of conjugacy classes,

ψ_1, ψ_2, ψ_3 are NOT injective on the set of conjugacy classes.

— THANK YOU —