On Logarithmic Knot Invariant

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Braids in Paris
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Contents:

1. Quantum $\mathcal{U}_q(sl_2)$ invariants
2. Volume conjecture
3. Logarithmic knot invariant
4. Logarithmic 3-manifold invariant

Slogan:

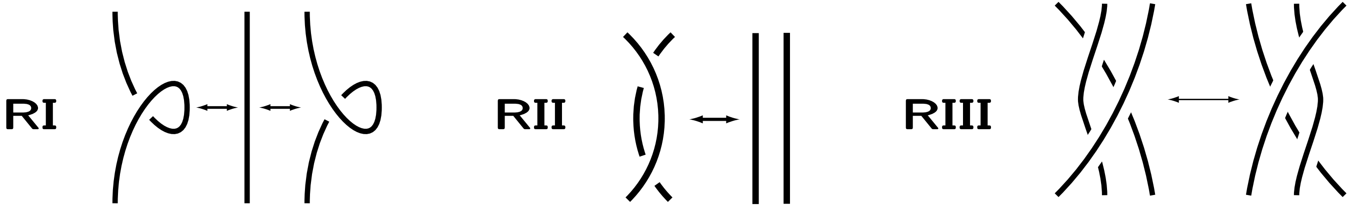
Step forward to **non-semisimple world**,
then we can see something new.

Especially around the *ridge* between semisimple and non-semisimple.

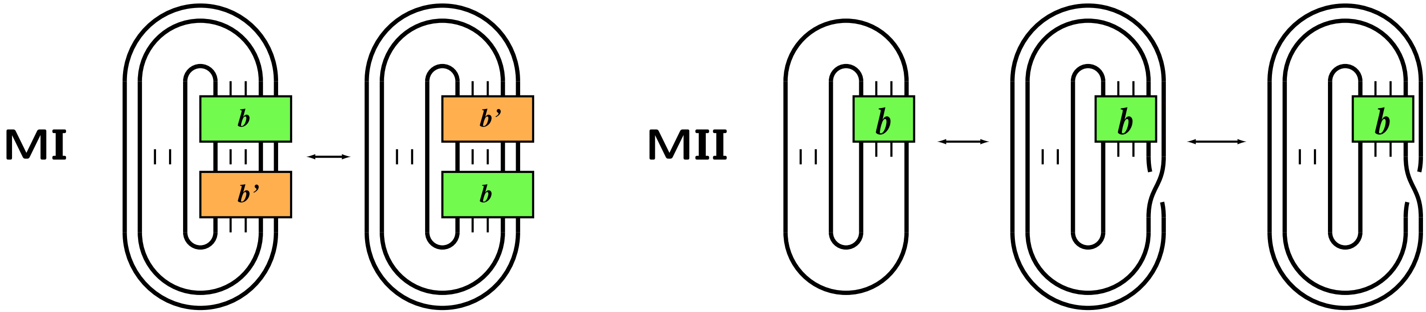
Contents:

Link invariant

- **Reidemeister moves**



- **Markov moves**



- **State model**

$\varphi : \{\text{parts of diagram}\} \to \text{a set}$ **state**

$$Z_L = \sum_{\varphi} \left(\prod_{m:\ \text{max, min}} W_m^{\varphi} \prod_{c:\ \text{crossing}} W_c^{\varphi} \right)$$ **partition function**

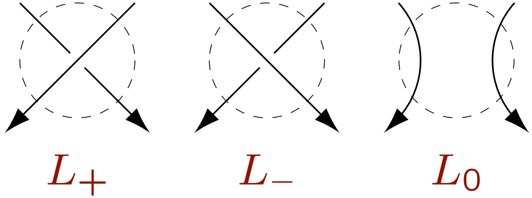
The Jones polynomial $V_L(q)$

- **Trace of the Temperley-Lieb algebra** $T_n(q)$
  from the theory of subfactors of a II_1 factor.

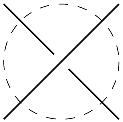
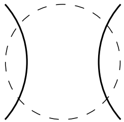
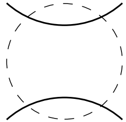
- **Skein relation**

$$q^{-2}\,V_{L_+}(q) - q^{2}\,V_{L_-}(q)$$

$$= (q - q^{-1})\,V_{L_0}(q).$$

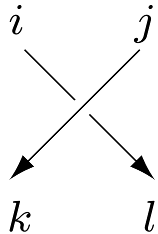


- **Kauffman bracket**

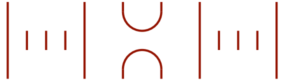
 $= q^{-1/2}$  $+ q^{1/2}$  .

- R**-matrix**

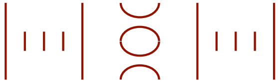
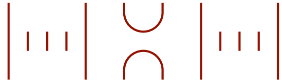
$$R = \begin{pmatrix} q^3 & 0 & 0 & 0 \\ 0 & q^3 - q & q^2 & 0 \\ 0 & q^2 & 0 & 0 \\ 0 & 0 & 0 & q^3 \end{pmatrix}$$

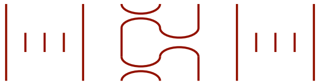
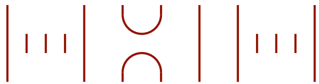


The Temperley-Lieb algebra $T_n(q)$

Generators $e_i =$ , $i = 1, 2, \cdots, n-1,$ $\quad \mathbf{1} =$ .

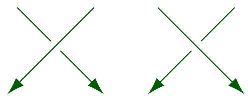
Relations $e_i{}^2 = -(q + q^{-1})\, e_i, \qquad e_i\, e_{i\pm 1}\, e_i = e_i.$

 $= -(q + q^{-1})$ ,

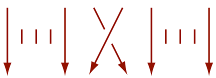
 $=$ .

- For generic q, $T_n(q)$ is semisimple.
- It is a quotient of the Iwahori-Hecke algebra, which is a deformation of the group ring of the sym. group S_n.
- It is also a quotient of the group ring of B_n.
- If $q = \exp(\frac{\pi i}{N})$ and $n \geq N$, then $T_n(q)$ is not semisimple.

Braid group representation

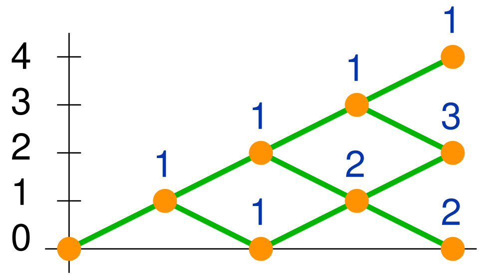


$+1 \quad -1$

$\sigma_i :$  $\longrightarrow q^{-1/2}\, \mathbf{1} + q^{1/2}\, e_i.$

Jones polynomial $\longleftarrow$ trace (with nomalization for *writhe*)

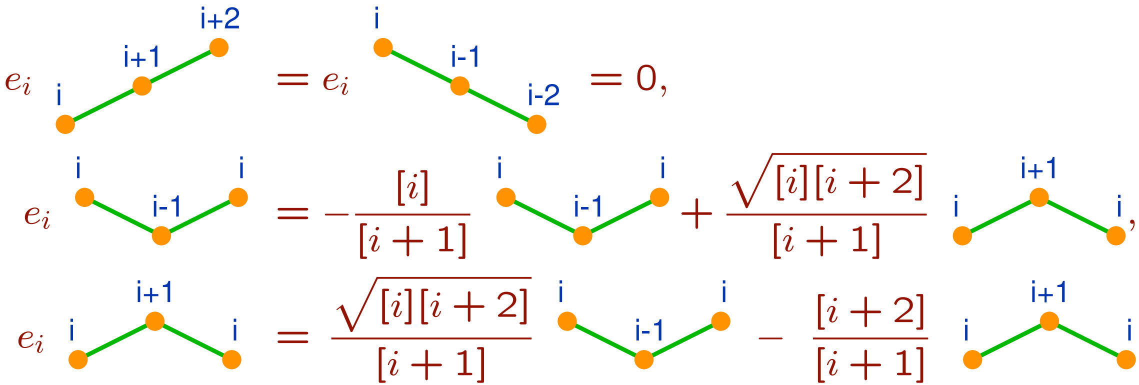
Representations of $T_n(q)$



Brattely diagram

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Action of $T_n(q)$ on paths of the Brattely diagram.



Regular representation

Basis: reduced words of $e_1, \cdots, e_{n-1}$.

I_j : two-sided ideal generated by $e_1 e_3 \cdots e_{2j-1}$.

$T_n(q) \cong I_{[j/2]} \oplus \cdots \oplus I_1/I_2 \oplus I_0/I_1$ simple algebras if q is generic

Remark. Path representation does not work well if q is a root of unity, while regular representation is well-defined even for such q.

Quantum group $\mathcal{U}_q(sl_2)$

$$\mathcal{U}_q(sl_2) = \langle K, E, F \mid \; [E,F] = \frac{K - K^{-1}}{q - q^{-1}},$$
$$K\,E\,K^{-1} = q^2\,E, \quad K\,F\,K^{-1} = q^{-2}\,F \rangle$$

Highest weight representation

Highest weight $\lambda \in \mathbb{Z}_{\geqq 0}$, U_λ: highest weight rep. of $\mathcal{U}_q(sl_2)$

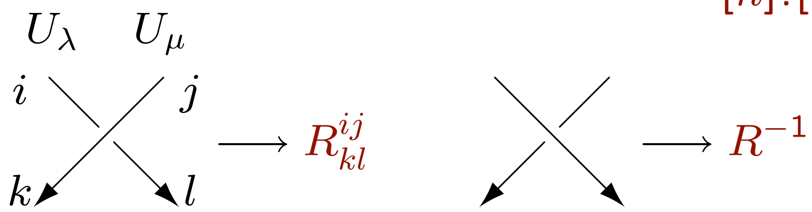
$$\begin{array}{cccccccccccc}
E & & & & \xleftarrow{[1]} & & \xleftarrow{[2]} & & & \xleftarrow{[2\lambda]} & & \\
K & 0 \longleftarrow & q^{\lambda} & & & q^{\lambda-1} & & & \cdots & & q^{-\lambda} & \\
 & & \diamondsuit_0 & & & \diamondsuit_1 & & \diamondsuit_2 & \cdots \; \diamondsuit_{\lambda-1} & & \diamondsuit_\lambda & \\
F & & & & \xrightarrow[{[2\lambda]}]{} & & \xrightarrow[{[2\lambda-1]}]{} & & \cdots & \xrightarrow[{[1]}]{} & & \longrightarrow 0
\end{array}$$

U_1 : 2-dim. repr., $\qquad$ U_λ is a direct summand of $\otimes^\lambda U_1$.

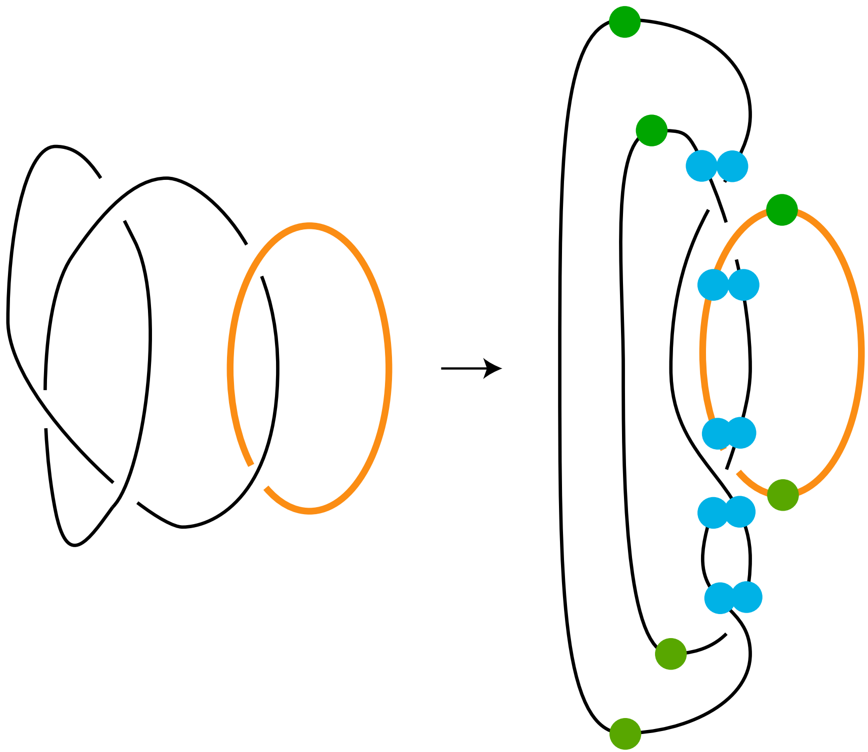
Representation of the R-matrix

$$[n]! = [n][n-1]\cdots[1]$$

$$R^{ij}_{kl} = \sum_n \delta_{l-i,n}\,\delta_{j-k,n}\,q^{\{(\lambda-2l)(\mu-2k)+n(n-1)\}/2}\,\frac{[l]![\mu-k]!}{[n]![i]![\mu-j]!}$$



Colored Jones invariant



$$\begin{array}{c}
\mathbb{C} \\
\downarrow \\
U_\lambda^* \otimes U_\lambda \\
\downarrow \\
U_\lambda^* \otimes U_\lambda^* \otimes U_\lambda \otimes U_\lambda \\
\downarrow R^{-1} \\
U_\lambda^* \otimes U_\lambda^* \otimes U_\lambda \otimes U_\lambda \\
\downarrow \\
U_\lambda^* \otimes U_\lambda^* \otimes U_\lambda \otimes U_\lambda \otimes U_\mu \otimes U_\mu^* \\
\downarrow R \\
U_\lambda^* \otimes U_\lambda^* \otimes U_\lambda \otimes U_\mu \otimes U_\lambda \otimes U_\mu^* \\
\downarrow R \\
U_\lambda^* \otimes U_\lambda^* \otimes U_\lambda \otimes U_\lambda \otimes U_\mu \otimes U_\mu^* \\
\downarrow \\
U_\lambda^* \otimes U_\lambda^* \otimes U_\lambda \otimes U_\lambda \\
\downarrow R^{-1} \\
U_\lambda^* \otimes U_\lambda^* \otimes U_\lambda \otimes U_\lambda \\
\downarrow R^{-1} \\
U_\lambda^* \otimes U_\lambda^* \otimes U_\lambda \otimes U_\lambda \\
\downarrow \\
U_\lambda^* \otimes U_\lambda \\
\downarrow \\
\mathbb{C}
\end{array}$$

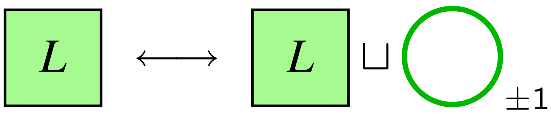
$\otimes^{2\lambda} U_1, \ \otimes^{2\mu} U_1$

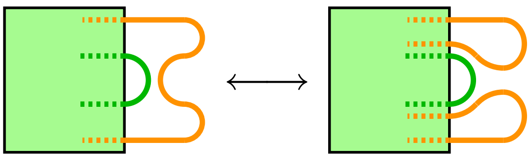
- The image of 1 is denoted by $V_L^{\lambda,\mu}(q)$.
- $\lambda = \mu = 1 \longrightarrow$ Jones polynomial
- Defined also from Jones poly. by *paralleling* the strings.

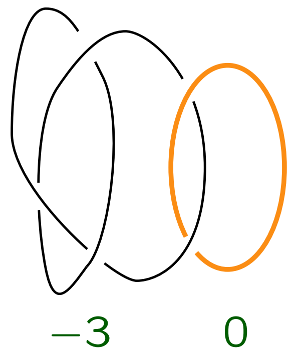
Witten-Reshetikhin-Turaev invariant

3-manifold $\longleftarrow$ surgery along a **framed link** in S^3

Two framed links L, L' represent the *same* manifold
$\Longleftrightarrow$ L' is obrained from L by a sequence of **Kirby moves**.

KI. 

KII. 



Let $q = e^{\pi i/N}$ and $\tau_N(M_L) = \sum_{\lambda_1,\cdots,\lambda_l=0}^{N-2} \prod_i [\lambda_i + 1] V_L^{\lambda_1,\cdots,\lambda_l}(q)$.

Then $\tau_N(M_L)$ is an invariant of the 3-manifold M_L.

Remark. Only use the restricted paths of the Brattely diagram. (stay in semisimple)

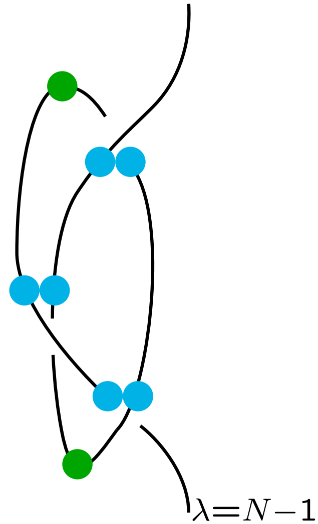
Contents:

Kashaev's conjecture

Kashaev's invariant

For a knot L, let $$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$K_N(L) = \left.\frac{V_L^{N-1}(q)}{[N]}\right|_{q=e^{\pi i/N}} = -i \sin\left(\frac{\pi}{N}\right) \frac{q}{N} \left.\frac{d\,V_L^{N-1}(q)}{dq}\right|_{q=e^{\pi i/N}}.$$

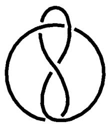
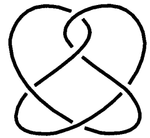
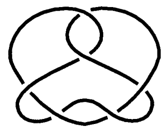


Conjecture (Kashaev)

For a hyperbolic knot L,

$$2\pi \log |K_N(L)| \underset{N\to\infty}{\sim} N \operatorname{Vol}(S^3 \setminus L).$$

Examples:

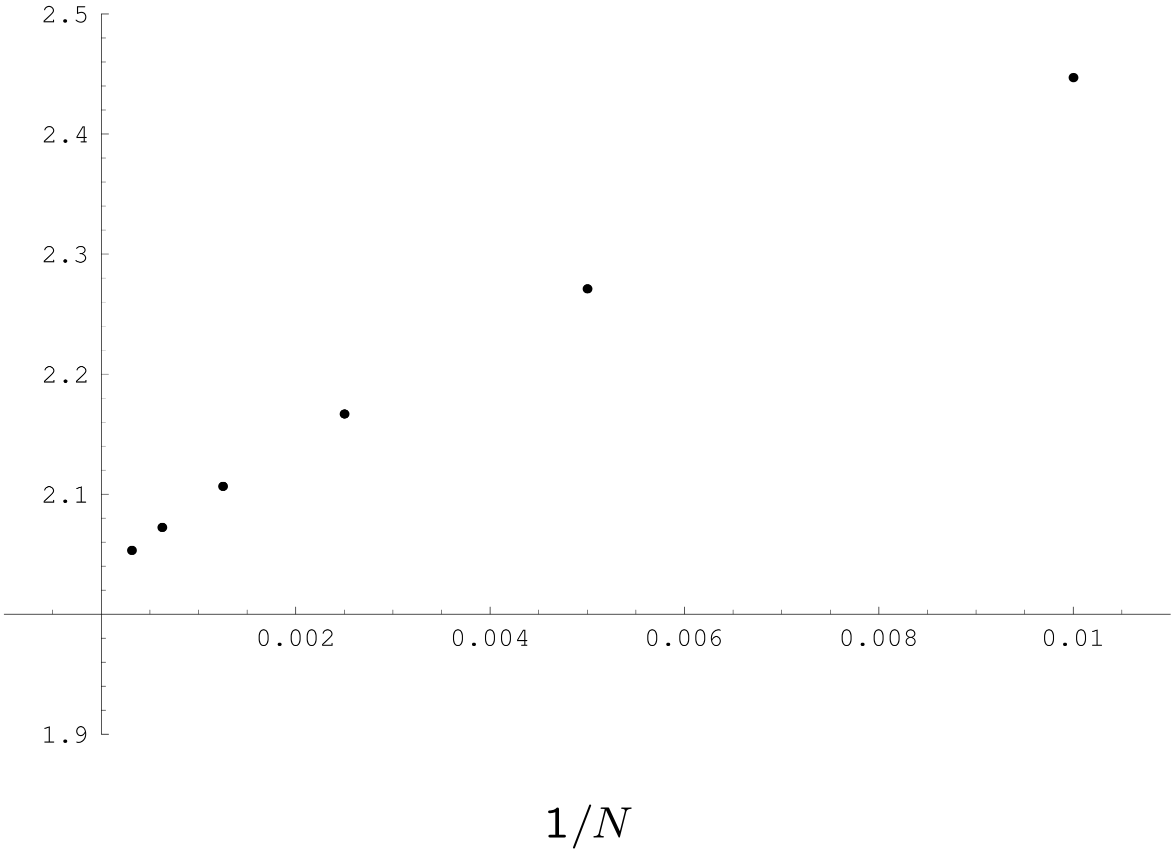
4_1 :  5_2 :  6_1 : 

$q = e^{\pi i/N}$, $(q)_k = \prod_{j=1}^{k}(1 - q^{2j})$,

$4_1 : \displaystyle\sum_{k=0}^{N-1} |(q)_k|^2$, $5_2 : \displaystyle\sum_{k\le l} q^{-k(l+1)} \frac{(q)_l^2}{(\bar{q})_k}$, $6_1 : \displaystyle\sum_{k+l\le m} q^{(m-k-l)(m-k+1)} \frac{|(q)_m|^2}{(q)_k(\bar{q})_l}$.

2.02988321 . . . 2.82812208 . . . 3.16396322 . . .

Kashaev's invariants for the figure-eight knot 4_1



Non-semisimplicity of Kashaev's invariant

The repsentation of $\mathcal{U}_q(sl_2)$ on the highest weight module U_{N-1} is irreducible even if $q = e^{\pi i/N}$. But $V_L^{N-1}(e^{\pi i/N}) = 0$ since the quantum dimension $[N] = 0$.

- The Temperley-Lieb algebra $T_n(e^{\pi i/N})$ is **not semisimple** and $K_N(L)$ is given by a **symmetric linear function** corresponding to the **radical** part of $T_n(e^{\pi i/N})$.

  Brattely diagram does not work for $K_N(L)$.

- The R-matrix is also **not semisimple**.

Example for $N = 2$.

- $q = i$, $-(q + q^{-1}) = 0$, $T_2(i) = \{a + b\, e_1\}$, ${e_1}^2 = 0$ and $K_2(L)$ is determined by the coefficient of e_1.

- $$R = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -2i & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \sim \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 1 & 0 \\ 0 & & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

  Jordan decomposition

Remarks for Kashaev's invariant

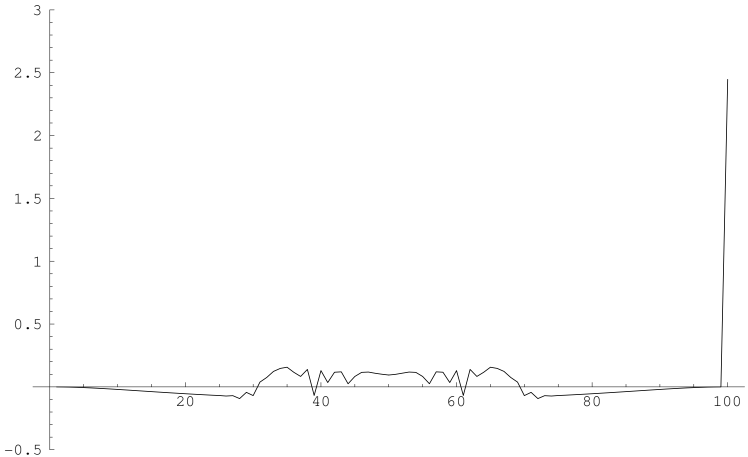
If $q = e^{\pi i/N}$, then the colored Jones invariants

$$\frac{V_L^0(q)}{[1]},\ \frac{V_L^1(q)}{[2]},\ \dots,\ \frac{V_L^\lambda(q)}{[\lambda+1]},\ \dots,\ \frac{V_L^{N-3}(q)}{[N-2]},\ \frac{V_L^{N-2}(q)}{[N-1]}$$

are much smaller than $K_N(L)$.

Colored Jones $\longrightarrow$ **polynomial growth**
Kashaev's invariant $\longrightarrow$ **exponential growth**

Colored Jones invariants for the figure-eight knot



$N = 100$

$\frac{2\pi \log}{N}$ of inv.

Yokota's theory

Kashaev's method

1. Replace $(q)_k$ by $\exp \mathrm{Li}_2(q^{2k})$. $\left(\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-x)}{x}\,dx\right)$

   Then get a function V consisting of Li_2 and $\log$, which is called the **potential function**. $(4_1 : V = \mathrm{Li}_2(z) - \mathrm{Li}_2(1/z))$

2. Replace the summation for the potential function V by an integral. $\left(4_1 : \int \exp \frac{Ni}{2\pi} (\mathrm{Li}_2(z) - \mathrm{Li}_2(1/z))\, dz\right)$

3. Apply the saddle point method to the integral**?**conjecture

   Saddle point $\longleftrightarrow$ $dV = 0$. $(4_1 : z^2 - z + 1 = 0)$

Then the value at the saddle point of V determines the asymptotics of $|K_N(L)|$. $(4_1 : \mathrm{Li}_2(e^{\pi i/3}) - \mathrm{Li}_2(e^{-\pi i/3}) = 2.02988\ldots)$

Theorem. (Y.Yokota) The **equation** $dV = 0$ coincides with the **hyperbolicity equation** for the simplicial decomposition of $S^3 \setminus L$ corresponding to the diagram of L.

Generalizations of the volume conjecture

- **Hyperbolic volume** (R.Kashaev) For a hyp. knot L,

$$2\pi \log |K_N(L)| \underset{N\to\infty}{\sim} N\,\mathrm{Vol}(S^3 \setminus L).$$

- **Simplicial volume** (H.M.-J.M.) For any knot L,

$$2\pi \log |K_N(L)| \underset{N\to\infty}{\sim} N\,\mathrm{Vol}_s(S^3 \setminus L)/c_3$$ (Vol_s simplicial volume)

- **Complex volume** (H.M.-J.M.-M.O.-T.T.-Y.Y.)

$$2\pi \log K_N(L) \underset{N\to\infty}{\sim} N\,\Big(\mathrm{Vol}(S^3 \setminus L) + i\,CS(S^3 \setminus L)\Big).$$

- **Closed manifold** (H.M.) (apply Kashev's method naively)

$$\underset{N\to\infty}{\text{o-lim}} \frac{2\pi \log \tau_N(L)}{M} = \mathrm{Vol}(M) + i\,CS(M).$$ (**optimistic limit**)

- **Asymptotic expansion conj.** (J. Andersen-S.Hansen)

  Actual asymptolics of $\tau_N(M)$

- **Deformation** (S. Gukov, H.M., Y.Y.)

  Deromation of $q \longrightarrow$ volume of cone manifolds

  Deformation of potential func. $\to$ A-polynomial of $S^3 \setminus L$

- **Volume of a tetrahedron** (J.M.-M.Y., J.M.-A.U.)

  Quantum $6j$-symbol $\underset{\text{o-lim}}{\longrightarrow}$ volume of tetrahedon (**optimistic limit**)

- **E.t.c.**

Problem

Construct a quantum 3-manifold invariant with **exponential** growth.

Contents:

Restricted (small) quantum group $\overline{\mathcal{U}}_q(sl_2)$

Definition

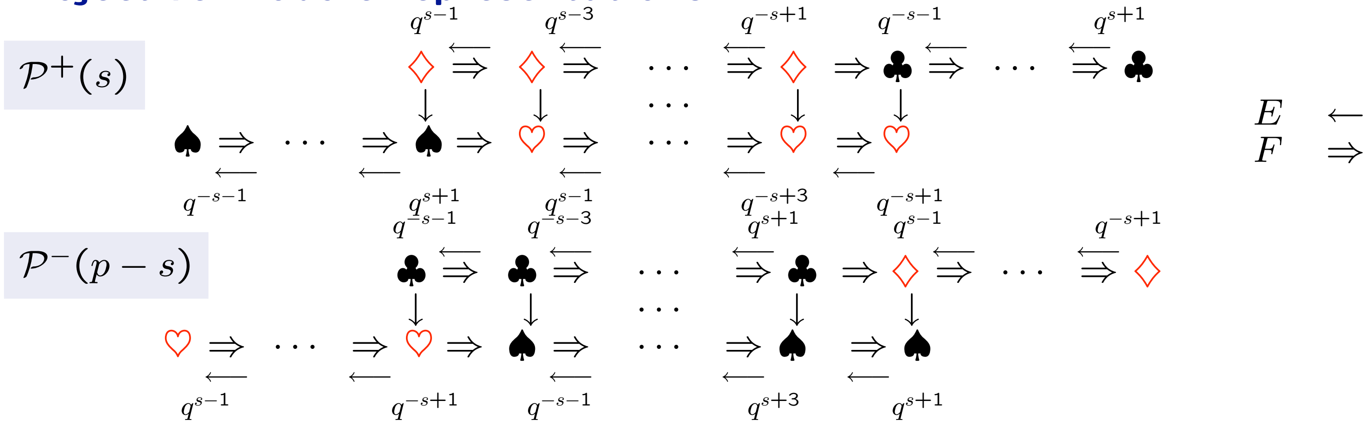
$q = e^{\pi i/N}$

$$\overline{\mathcal{U}}_q(sl_2) = \left\langle K, E, F \mid \ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \right.$$

$$\left. E^p = F^p = 0, \ K^{2p} = 1, \ K\,E\,K^{-1} = q^2\,E, \ K\,F\,K^{-1} = q^{-2}\,F \right\rangle$$

Basis $\{E^i K^j F^k : 0 \le i, k \le N-1, \ 0 \le j \le 2N-1\}$ $2\,N^3$-dimensional.

Projective module representations



References:

- **N.Reshetikhin, V.G.Turaev:** Invariants of 3-mfds via link polynomials and quantum groups, **Invent. math. 103 (1991), 547–597.**
- **M.Jimbo, T.Miwa, Y.Takeyama:** Counting minimal form factors of ..., **Mosc. Math. J. 4 (2004), 787–846, 981.**

Representations

$\mathcal{P}^+(s)$: diagonal parts $U_{s-1} \oplus U_{-s-1} \oplus U'_{-s-1} \oplus U'_{s-1}$

$\mathcal{P}^-(s)$: diagonal parts $U_{-s-1} \oplus U_{s-1} \oplus U'_{s-1} \oplus U'_{s-1}$

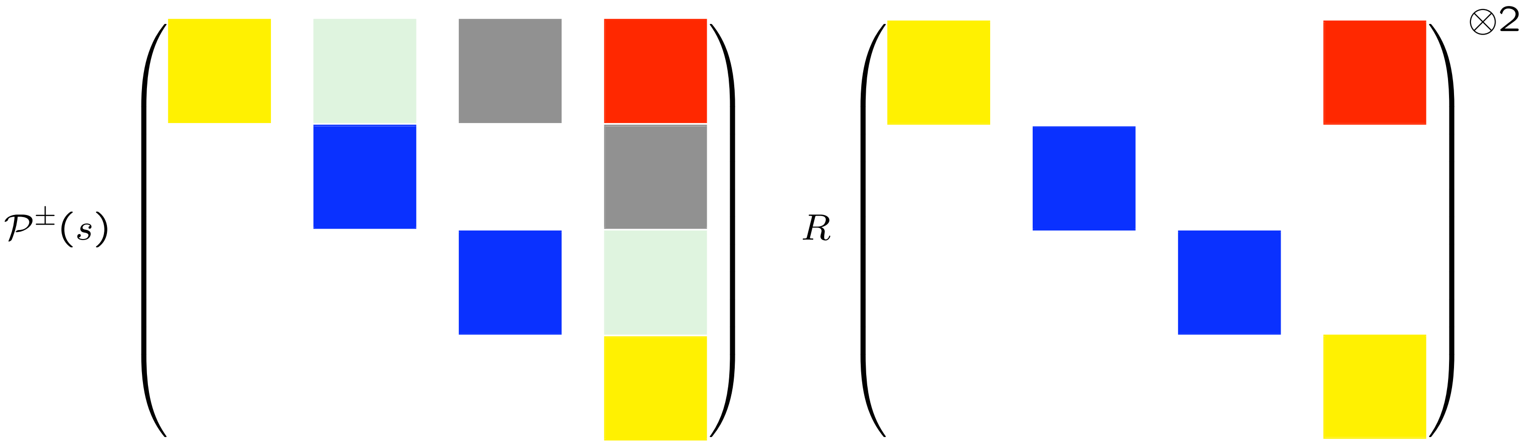
$\mathcal{P}^+(0)$: N-dim. representation of highest weight -1

$\mathcal{P}^-(0)$: N-dim. representation of highest weight $N-1$

$\Longleftrightarrow$ Kashaev's invariant corresponding to the volume

The **universal R-matrix** acts on $\mathcal{P}^\pm(s) \otimes \mathcal{P}^\pm(r)$.

The diagonal part is a direct sum of the usual R-matrix, and it has off-diagonal elements sending $U_{\pm s-1}$ to $U'_{\pm s-1}$ and/or $U_{\pm r-1}$ to $U'_{\pm r-1}$.



Center of $\overline{\mathcal{U}}_q(sl_2)$

The action of the Casimir element on $\mathcal{P}^+(s)$ and $\mathcal{P}^-(N-s)$ are the same. (act by the same scalar)

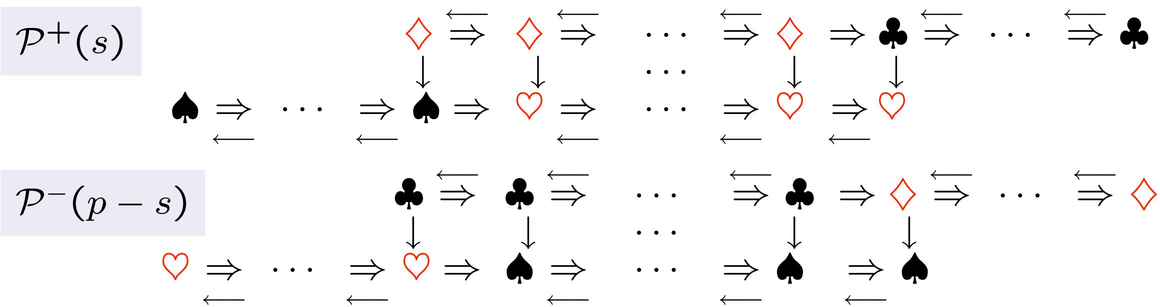
Basis of the center (e_i : principal idempotent):

e_s : acts on $\mathcal{P}^+(s)$ and $\mathcal{P}^-(N-s)$ as the id. $\quad (1 \le s \le N-1)$

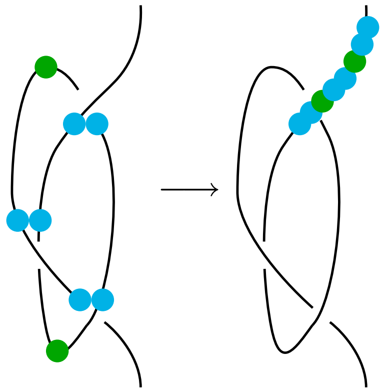
e_0 : acts on $\mathcal{P}^+(0)$ as id. $\quad w_s^+$: maps $\diamondsuit$ of $\mathcal{P}^+(s)$ to $\heartsuit$

e_N : acts on $\mathcal{P}^-(0)$ as id. $\quad w_s^-$: maps $\clubsuit$ of $\mathcal{P}^-(s)$ to $\spadesuit$ $\quad (1 \le s \le N-1)$

$$\overset{e_s}{\begin{pmatrix} E_s & & & \\ & E_{N-s} & & \\ & & E_{N-s} & \\ & & & E_s \end{pmatrix}} \quad \overset{w_s^+}{\begin{pmatrix} O & & & E_s \\ & O & & \\ & & O & \\ & & & O \end{pmatrix}} \begin{matrix} \heartsuit \\ \clubsuit \\ \spadesuit \\ \diamondsuit \end{matrix} \quad \overset{w_{N-s}^-}{\begin{pmatrix} O & & & E_{N-s} \\ & O & & \\ & & O & \\ & & & O \end{pmatrix}} \begin{matrix} \spadesuit \\ \diamondsuit \\ \heartsuit \\ \clubsuit \end{matrix}$$



Tangle to center



Knot $L \to$ (1,1)-tangle $T \to$ center $\mathcal{Z}$ of $\overline{\mathcal{U}}_q(sl_2)$

$$T \mapsto z_T = \sum_{s=0}^{N} a_s\, e_s + \sum_{s=1}^{N-1} \left(b_s^+\, w_s^+ + b_s^-\, w_{N-s}^- \right)$$

a_s : colored Jones invariant (from diagonal part)

$b_s^\pm$: **logarithmic knot invariant** (from radical part)

Remark. For a link L, they may depend on the opened component.

The colored Alexander invariant $L = L_1 \cup \cdots \cup L_k$

Defined from the R-matrix of the N-dimensional simple highest weight module of $\mathcal{U}_q(sl_2)$ at $q = \exp \pi i/N$ with a generic highest weight λ.

$O_\lambda^N(T)$: the scalar corresponding to T. (quantum dimension $= 0$)

$$\Phi_\lambda^N(L) = O_\lambda^N(T)\, \frac{\sin \frac{\pi(\lambda_1+1)}{N}}{\sin \pi(\lambda_1+1)}$$

normalized for the opened component L_1.

Y. Akutsu, T. Deguchi, T. Ohtsuki: Invariants of colored links,
J. Knot Theory Ramifications 1 (1992), no. 2, 161–184.

Relation to the colored Alexander invariant

Theorem. (J.M.-K.Nagatomo)

For a knot L and an integer $1 \leq s \leq N-1$,

$$b_s^+(L) = \text{const.} \lim_{\varepsilon \to 0} \left(\Phi^N_{s-1+\varepsilon}(L) + \Phi^N_{-s-1+\varepsilon}(L) \right),$$

$$b_s^-(L) = \text{const.} \lim_{\varepsilon \to 0} \left(\Phi^N_{s-1+\varepsilon}(L) + \Phi^N_{2N-s-1+\varepsilon}(L) \right).$$

Properties

1. Grows **exponentially** for a hyperbolic knot.
2. $b_s^+(L) - b_s^-(L) = \text{const.}\, f_L\, V_L^{s-1}(q)$,
   which grows polynomially. Here f_L is the framing of L .
3. If L is a split link, then these invariants vanish.

---

colored Jones	Kashaev's invariant	colored Alexander
q : generic ↙	$q = \exp \frac{\pi i}{N}$, $\lambda = N-1$	↘ λ : generic
semisimple r-matrix	non-semisimple r-matrix	semisimple r-matrix
	λ : integer ↙	
	logarithmic invariant	

Comparison between invariants of 4_1

$$q = e^{\pi i/N}, \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [n][n-1]\cdots[1], \quad [0]! = 1.$$

Colored Jones invariant λ **: half integer**

$$\sum_{j=0}^{\lambda} (q - q^{-1})^{2j} \frac{[\lambda + j + 1]!}{[\lambda - j]!} = \sum_{j=0}^{\max(\lambda, N-2-\lambda)} (-4)^j \left(\sin\frac{\pi}{N}\right)^{2j} \frac{[\lambda + j + 1]!}{[\lambda - j]!}$$

oscillating

Kashaev's invariant

$$\sum_{j=0}^{N-1} (q - q^{-1})^{2j} \frac{[N + j]!}{[N][N - j - 1]!} = \sum_{j=0}^{N-1} 4^j \left(\sin\frac{\pi}{N}\right)^{2j} ([j]!)^2$$

positive

Colored Alexander invariant λ **: generic**

$$\sum_{j=0}^{N-1} \frac{(q - q^{-1})^{2j}}{2\, i^N \sin(\lambda\pi)} \frac{[\lambda + j + 1]!}{[\lambda - j]!} = \sum_{j=0}^{N-1} \frac{(-4)^j\, (\sin \pi/N)^{2j}}{2\, i^N \sin(\lambda\pi)} \frac{[\lambda + j + 1]!}{[\lambda - j]!}$$

Logarithmic invariant b_s^+ $(= b_s^-)$ $s \sim \lambda + 1$ **: integer**

$$\text{const.} \sum_{j=\min(s, N-s)}^{\max(s, N-s)} (-4)^j \left(\sin\frac{\pi}{N}\right)^{2j} \frac{[s + j]!}{[N][s - j - 1]!}$$

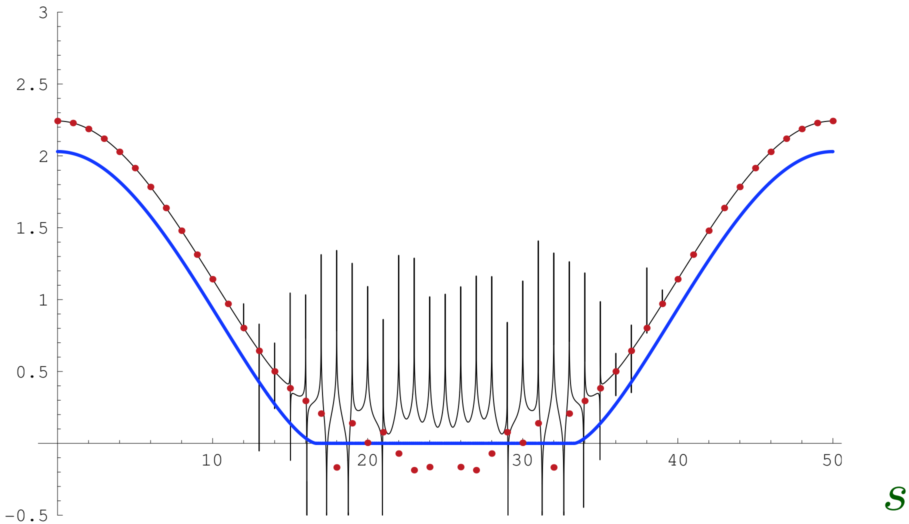
positive

Let s to be N, then it becomes Kashaev's invariant.

Actual values for the figure-eight knot

$N = 50$

$$\log \frac{2\pi|\bullet|}{N}$$

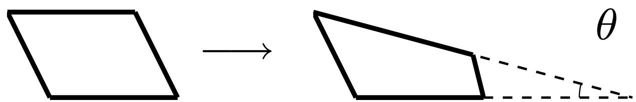


$\bullet$: logarithmic knot invariants b_s^+

— : colored Alexander invariant

— : volume of the cone manifold with cone angle $2(N-s)\pi/N$

Cone manifold. Cusp diagram



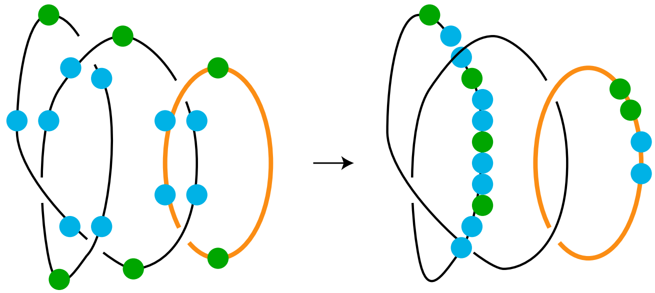
Contents:

Universal invariant of $\overline{\mathcal{U}}_q(sl_2)$ R. Lawrence, T. Ohtsuki

Construction

$L = L_1 \cup \cdots \cup L_k$: framed link

$u_1 \otimes u_2 \otimes \cdots \otimes u_k \in \overline{\mathcal{U}}_q(sl_2) \otimes \overline{\mathcal{U}}_q(sl_2) \otimes \cdots \otimes \overline{\mathcal{U}}_q(sl_2).$



Beads construction

$$f_i \in \left(\overline{\mathcal{U}}_q(sl_2) \big/ [\overline{\mathcal{U}}_q(sl_2), \overline{\mathcal{U}}_q(sl_2)] \right)^*$$

space of **symmetric linear functions**

$\prod_{i=1}^{k} f_i(u_i)$: inv. of L.

- **Colored Jones invariant** : f_i is the trace on U_λ.
- **Hennings invariant** : $f_i = \phi$ (modified right integral)

$\lambda : \overline{\mathcal{U}}_q(sl_2) \longrightarrow \mathbb{C}$ is the **right integral** of $\overline{\mathcal{U}}_q(sl_2)$ as a finite dimensional Hopf algebra if it satisfies

$$\lambda(x)\,1 = m(\lambda \otimes 1)(\Delta(x)), \quad \lambda(x\,y) = \lambda(S^2(y)\,x).$$

Let $\phi(x) = \lambda(K^{p+1}\,x)$. Then $\phi(x\,y) = \phi(y\,x)$ and ϕ is a simmetric linear function of $\overline{\mathcal{U}}_q(sl_2)$.

Hennings invariant

Theorem (Hennings)

$\phi(u_1)\phi(u_2)\cdots\phi(u_k)$ is an invariant of 3-manifold obtained by the surgery along K.

References:

- **G.Kuperberg:** Involutory Hopf algebras and 3-manifold invariants. **Internat. J. Math. 2 (1991), 41–66.**
- **M.Hennings:** Invariants of links and 3-manifolds obtained from Hopf algebras. **J. London Math. Soc. (2) 54 (1996), 594–624.**
- **L.Kauffman, D.Radford:** Invariants of 3-manifolds derived from finite-dimensional Hopf algebras. **J. Knot Theory Ramifications 4 (1995), 131–162.**
- **T.Ohtsuki:** Invariants of 3-mfds derived from universal invariants of framed links. **Math.Proc.Camb. 117 (1995), 259–273.**
- **Q.Chen, S.Kuppum, P.Srinivasan:** On the relation between the WRT invariant and the Hennings invariant, **arXiv:0709.2318.**

Theorem (Q.Chen, S.Kuppum, P.Srinivasan)

$$\phi(u_1)\phi(u_2)\cdots\phi(u_k) = |H_1(M_K)|\ \tau_p(M_K)$$

Remark. The Hennings invariant grows **polynomially**.

General principle

If there exist **good S-matrix**, we can make **everything**, including CFT, TQFT, knot invariant, 3-manifold invariant, e.t.c.

Good S-matrix is given in the last reference.

References:

- **G.Moore, N.Seiberg:** Classical and quantum conformal field thoery, **Comm. Math. Phys. 123 (1989), 177–254.**
- **V.Lyubashenko:** Modular transformations for tensor categories, **J. Pure Appl. Algebra 98 (1995), 279–327.**
- **V.Lyubashenko, S.Majid:** Braided groups and quantum Fourier transform, **J. Algebra 166 (1994), 506–528.**
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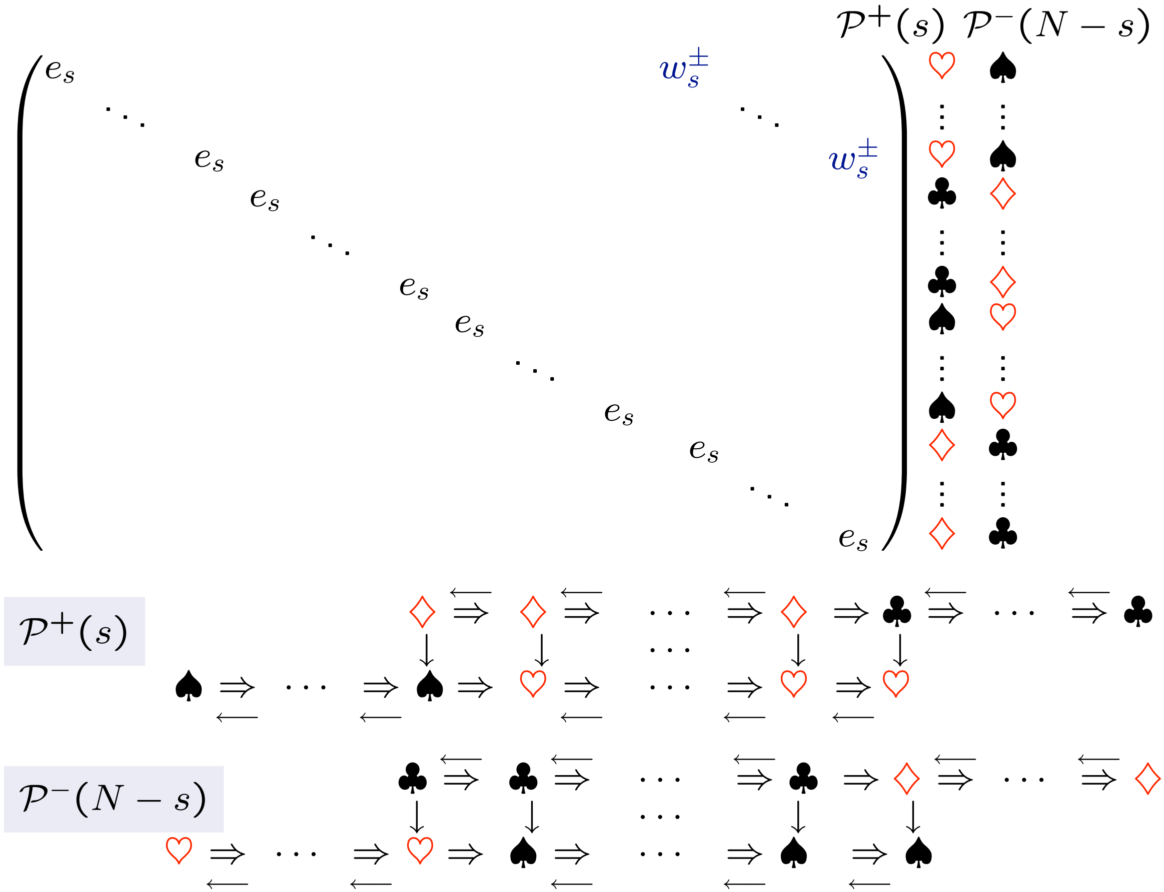
- **T.Kerler:** Mapping class group actions on quantum doubles, **Comm. Math. Phys. 168 (1995), 353–388.**
- **B.Feigin, A.Gainutdinov, A.Semikhatov, I.Tipunin:** Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center, **Comm. Math. Phys. 265 (2006), 47–93.**

Center of $\overline{U}_q(sl_2)$

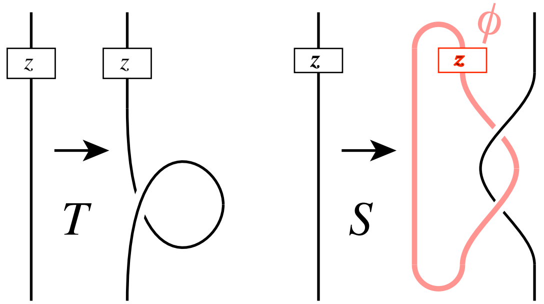


$SL(2,\mathbb{Z})$ action on center $\mathcal{Z} \subset \overline{\mathcal{U}}_q(sl_2)$

generators

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



Good basis of the center of $\overline{\mathcal{U}}_q(sl_2)$

$$[s] = \frac{q^s - q^{-s}}{q - q^{-1}}$$

$$\widehat{\kappa}(0) = (-1)^{N+1} e_0$$

$$\widehat{\kappa}(s) = \frac{1}{[s]^2} \left(w_s^+ + w_{N-s}^-\right)$$

$$\widehat{\kappa}(N) = e_N$$

$$\widehat{\rho}(s) = (-1)^{N+s} \frac{1}{N\,(q^s - q^{-s})} \left(e_s - \frac{q^s + q^{-s}}{[s]^2} (w_s^+ + w_{N-s}^-)\right)$$

$$\widehat{\varphi}(s) = \frac{1}{[s]^2} \left(\frac{N-s}{N} w_s^+ - \frac{s}{N} w_{N-s}^-\right) \qquad (1 \le s \le N-1)$$

Structure of $SL(2,\mathbb{Z})$ representation

(B.Feigin, A.Gainutdinov, A.Semikhatov, I.Tipunin)

It comes from the logarithmic Conformal Field Theory.

$$\widehat{\kappa}(s), \quad (\widehat{\rho}(s), \widehat{\varphi}(s))$$

$$\mathcal{Z} \cong \mathbb{C}^{N+1} \oplus \left(\mathbb{C}^2 \otimes \mathbb{C}^{N-1}\right)$$

Volume H_1, WRT **Hennings** $\lambda_{N,s} = q^{\frac{s^2-1}{2}}$

$$T\,\widehat{\kappa}(s) = \lambda_{N,s}\,\widehat{\kappa}(s),\ T\,\widehat{\varphi}(s) = \lambda_{N,s}\,\widehat{\varphi}(s),\ T\,\widehat{\rho}(s) = \lambda_{N,s}\,(\widehat{\rho}(s) + \widehat{\varphi}(s)).$$

$$S\,\widehat{\kappa}(s) = \frac{1}{\sqrt{2N}}\left((-1)^{N-s}\widehat{\kappa}(0) + \sum_{t=1}^{N-1}(-1)^{N+t+s}(q^{st} + q^{-st})\widehat{\kappa}(t) + \widehat{\kappa}(N)\right),$$

$$S\,\widehat{\rho}(s) = \frac{1}{\sqrt{2N}}\sum_{t=1}^{N-1}(-1)^{N+t+s}(q^{st} - q^{-st})\widehat{\varphi}(t),$$

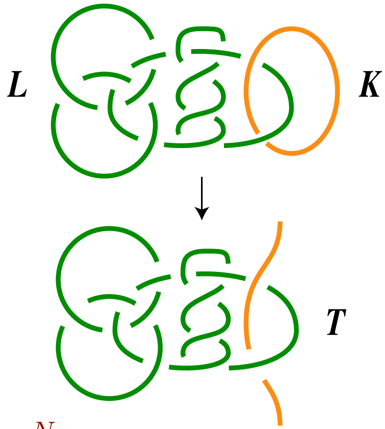
$$S\,\widehat{\varphi}(s) = \frac{1}{\sqrt{2N}}\sum_{t=1}^{N-1}(-1)^{N+t+s}(q^{st} - q^{-st})\widehat{\rho}(t).$$

$$\widehat{\kappa}(0) = (-1)^{N+1}\,e_0, \quad \widehat{\kappa}(s) = \frac{1}{[s]^2}\left(w_s^+ + w_{N-s}^-\right), \quad \widehat{\kappa}(N) = e_N$$

$$\widehat{\rho}(s) = \frac{(-1)^{N+s}}{N\,(q^s - q^{-s})}\left(e_s - \frac{q^s + q^{-s}}{[s]^2}\,(w_s^+ + w_{N-s}^-)\right), \quad \widehat{\varphi}(s) = \frac{1}{[s]^2}\left(\frac{N-s}{N}\,w_s^+ - \frac{s}{N}\,w_{N-s}^-\right)$$

Logarithmic invariant of knots in 3-manifolds

Let $L \cup K$ be a framed link in S^3, where L be a link and K be a knot. Let M be a 3-manifold obtained from L. Then K is considered as a knot in M. Let T be a tangle obtained by cutting on K, then there is an element $z_T \in \mathcal{Z}$ (center of $\mathcal{U}_q(sl_2)$) corresponding to the tangle T obtained by applying ϕ to the components of L.



$$z_T = \sum_{s=1}^{N-1} \Big(a_s(T)\,\widehat{\rho}(s) + b_s(T)\,\widehat{\phi}(s) \Big) + \sum_{s=0}^{N} c_s(T)\,\widehat{\kappa}(s).$$

Theorem The element of the center z_T is an invariant of the knot K in M and the coefficients $a_s(T)$, $b_s(T)$ and $c_s(T)$ are also invariants of K.

Remark. 1. $c_s(T)$ grow **exponentially** ($a_s(T)$, $b_s(T)$ grow polynomially.)

2. Applying ϕ to z_T, we get the Hennings invariant of M_K, which is a linear combination of $b_s(T)$.

Logarithmic link invariant

Definition. $\Phi^N_\lambda(L)$: colored Alexander invariant

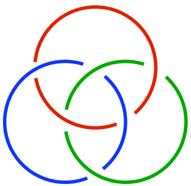
Let $L = L_1 \cup L_2 \cup \cdots \cup L_k$ be a k-component link. For an multi-indices $\mathbf{s} = (s_1, \cdots, s_k)$ $(0 \le s_j \le N-1)$,

$$\Psi^N_{\mathbf{s}}(L) = \text{const.} \lim_{\varepsilon \to 0} \left(\Phi^N_{\mathbf{s}-1+\varepsilon}(L) + \Phi^N_{-\mathbf{s}-1+\varepsilon}(L) \right).$$

Example. Borromean ring L

$$\Phi^N_{\lambda,\mu,\nu}(L) = \frac{i^N}{8 \sin \pi\lambda \sin \pi\mu \sin \pi\nu} \sum_{[N/2] \le s \le N-1} (-1)^{s+1} (q - q^{-1})^{2N+4s+1} \frac{[N]!^2\,[s]!^2}{[2s+1]!^2} \frac{[\lambda+s+1]!}{[\lambda-s]!} \frac{[\mu+s+1]!}{[\mu-s]!} \frac{[\nu+s+1]!}{[\nu-s]!}$$

$$\Psi^N_{p.q.r}(L) = \text{const.} \sum_{[N/2] \le s \le N-1} (-1)^{p+q+r+s+1} (q - q^{-1})^{2N+4s+1} \frac{[N]!^2\,[s]!^2}{[2s+1]!^2} \frac{[p+s+1]!}{[N]\,[p-s]!} \frac{[q+s+1]!}{[N]\,[q-s]!} \frac{[r+s+1]!}{[N]\,[r-s]!}$$



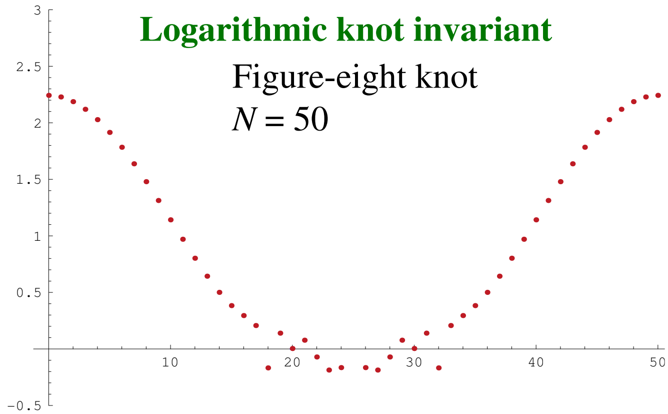
Note. The signatures do not change.

3-manifold invariant with exponential growth (?)

$L = L_1 \cup \cdots \cup L_k$: framed link $\quad M_L$ corresponding 3-manifold

Definition.

$$T_N(L) = \sum_{s_1,\cdots,s_k=0}^{N} \left(\prod_j (-1)^j \, [s_j] \right) \Psi^N_{s_1,\cdots,s_k}(L).$$



PROBLEM

1. Express $T_N(L)$ in terms of logarithmic TQFT.
   $T_N(L)$ grows exponentially and it should relate to the $\widehat{\kappa}$ part of the TQFT.
2. Is $T_N(L)$ an invariant of M_L? If not, is it possible to modify $T_N(L)$ to get an invariant of M_L?
3. Is the following true? (**Volume conjecture**)

$$2\pi \log T_N(L) \underset{N\to\infty}{\sim} N \left(\mathrm{Vol}(M_L) + i\, CS(M_L)\right).$$

Conclusion

- The logarithmic invariant, the colored Alexander invariant and the colored Jones invariant are given by similar formulas. The essential difference is the range of parameters.

- The representations $\mathcal{P}^+(s)$, $\mathcal{P}^-(s)$ $(s = 0, 1, \cdots, N-1)$ form a tensor category which does not have a unit object.

- The logarithmic invariant of a hyperbolic manifold grows exponentially while the Witten-Reshetikhin-Turaev invariant grows polynomially.

- In the logarithmic TQFT, the exponential growth part lives in the radical part.

- Standard tools do not work well for the exponential growth part. $\longrightarrow$ difficulty of quantum volume

  standard tools: semisimplicity, unitarity, rigidity, trace, ...

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Merci beaucoup!