A representation of the braid group from Lefschetz fibrations

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Outline of Part I

Outline of Part II

Part I

A Representation of the Braid Group

The (framed) braid group

Z finite set $\subset \mathbb{D} = \{z \in \mathbb{C} \ : \ |z| \leq 1\}$, $z_0 \in \partial\mathbb{D}$, $m = \mathrm{card}(Z)$

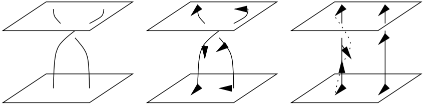
Definition (braid group)

$$\mathcal{B} := \pi_1(\mathrm{Sym}_m(\mathring{\mathbb{D}}) \setminus \Delta, [Z])$$

Choose *nonzero* tangent vectors $v_z \in T_z\mathbb{D}$, $z \in Z$

Definition (framed braid group)

$$\widetilde{\mathcal{B}} = \pi_1(\mathrm{Sym}_m(T\mathring{\mathbb{D}} \setminus 0) \setminus \Delta, [Z, \{v_z\}])$$



Mapping class groups

$$\begin{aligned} \mathcal{G} &:= \{\phi \in \mathrm{Diff}_0(\mathbb{D}, z_0) \ : \ \phi(Z) = Z\} \\ \mathcal{G}_0 &:= \{\phi \in \mathcal{G} \ : \ \exists \ \phi_t \in \mathcal{G} \text{ s.t. } \phi_0 = \mathrm{Id}, \ \phi_1 = \phi\} \end{aligned}$$

Proposition

There is an isomorphism $\mathcal{G}/\mathcal{G}_0 \simeq \mathcal{B}$.

$$\begin{aligned} \widetilde{\mathcal{G}} &:= \{\phi \in \mathrm{Diff}_0(\mathbb{D}, z_0) \ : \ \phi(Z) = Z, \ d\phi(z)v_z = v_{\phi(z)}, \ \forall z \in Z\} \\ \widetilde{\mathcal{G}}_0 &:= \{\phi \in \widetilde{\mathcal{G}} \ : \ \exists \phi_t \in \widetilde{\mathcal{G}} \text{ s.t. } \phi_0 = \mathrm{Id}, \ \phi_1 = \phi\} \end{aligned}$$

Proposition

There is an isomorphism $\widetilde{\mathcal{G}}/\widetilde{\mathcal{G}}_0 \simeq \widetilde{\mathcal{B}}$.

Distinguished configurations

Recall the choice of Z, z_0, $\{v_z\}_{z \in Z}$, and $m := \mathrm{card}(Z)$

Definition

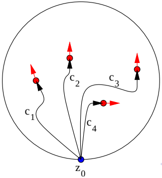
A *(marked) distinguished configuration* is an m-tuple $c = (c_1, \dots, c_m)$ of smooth embedded paths $c_i : [0,1] \to \mathbb{D}$ s.t.

i. $c_i(0) = z_0$, and c_i intersects c_j only at z_0 for all $i \neq j$;

ii. $\{c_1(1), \dots, c_m(1)\} = Z$;

iii. the vectors $\dot{c}_1(0), \dots, \dot{c}_m(0)$ are pairwise non-collinear and ordered clockwise at $T_{z_0}\mathbb{D}$;

iv. *(marked)* $\dot{c}_i(1) = -v_{c_i(1)}$.



Distinguished configurations

$\mathcal{C} = \{$ homotopy classes of distinguished configurations $\}$

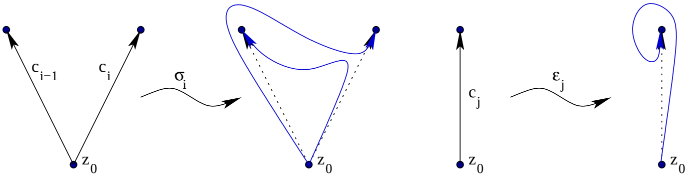
Proposition

The braid group $\mathcal{G}/\mathcal{G}_0$ acts freely and transitively on $\mathcal{C}$.

$\widetilde{\mathcal{C}} = \{$ homotopy classes of *marked* distinguished configurations $\}$

Proposition

The framed braid group $\widetilde{\mathcal{G}}/\widetilde{\mathcal{G}}_0$ acts freely and transitively on $\widetilde{\mathcal{C}}$.



Generators and relations

$$\mathcal{B}_m := \left\langle \sigma_2, \ldots, \sigma_m \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i, \ |i-j| \geq 2 \end{array} \right\rangle$$

$$\widetilde{\mathcal{B}}_m := \left\langle \begin{array}{l} \sigma_2, \ldots, \sigma_m \\ \varepsilon_1, \ldots, \varepsilon_m \end{array} \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \varepsilon_i\sigma_i = \sigma_i\varepsilon_{i-1} \\ \varepsilon_{i-1}\sigma_i = \sigma_i\varepsilon_i \\ \text{other generators commute} \end{array} \right\rangle$$

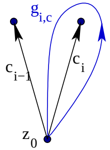
Proposition

$$\mathcal{B}_m \simeq \mathcal{B}, \qquad \widetilde{\mathcal{B}}_m \simeq \widetilde{\mathcal{B}}.$$

The above isomorphisms depend on the choice of a (marked) distinguished configuration c!

The Monodromy cocycle

Fix $c \in \widetilde{\mathcal{C}}$ $\rightsquigarrow$ ordering $Z = \{z_1, \ldots, z_m\}$



distinguished loops $g_{i,c} \in \Gamma = \pi_1(\mathbb{D} \setminus Z; z_0)$

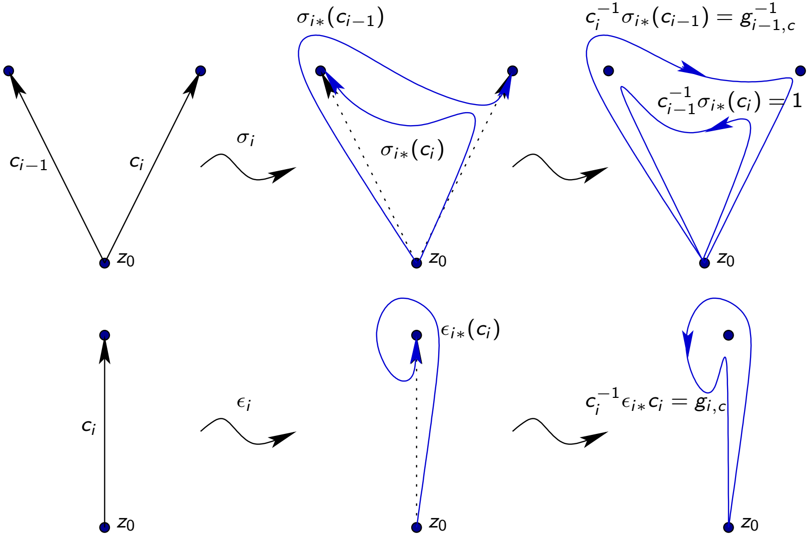
$\sigma \in \widetilde{\mathcal{B}}$ $\rightsquigarrow$ permutation $\pi_{\sigma,c} \in \mathfrak{S}_m$ via $\sigma(z_i) = z_{\pi_{\sigma,c}(i)}$

Definition (Massuyeau, O., Salamon)

Define $S_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ by

$$S_c(\sigma)_{ij} := \begin{cases} c_i^{-1} \cdot \sigma_* c_j, & i = \pi_{\sigma,c}(j), \\ 0, & \text{otherwise.} \end{cases}$$

The Monodromy cocycle



Point of view: non-abelian cohomology

Definition

Let G and A be (non-abelian) groups, such that G acts on A on the left $(g, a) \mapsto g_* a$. A map $s : G \to A$ is a 1-*cocycle* if

$$s(gh) = s(g) g_* s(h).$$

Two cocycles $s_0, s_1 : G \to A$ are *cohomologous* if $\exists a \in A$ s.t.

$$s_1(g) = a^{-1} s_0(g) g_* a.$$

Example: $\widetilde{\mathcal{B}}$ acts on Γ, hence on $\mathrm{GL}_m(\mathbb{Z}[\Gamma])$ componentwise

Proposition

(i) Each map $S_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ *is a 1-cocycle.*
(ii) Any two cocycles S_c *and* $S_{\tau_* c}$ *are cohomologous:*

$$S_{\tau_* c}(\sigma) = S_c(\tau)^{-1} S_c(\sigma) \sigma_* S_c(\tau), \quad \forall \sigma \in \widetilde{\mathcal{B}}.$$

$[S_c]$ *is canonically defined: the "Lefschetz monodromy class"*

Proof of the cocycle properties

(i) We prove $S_c(\sigma\tau) = S_c(\sigma)\sigma_* S_c(\tau)$.
Set $j := \pi_{\tau,c}(k)$, $i := \pi_{\sigma,c}(j) = \pi_{\sigma\tau,c}(k)$. Then

$$c_i^{-1} \cdot \sigma_* c_j \cdot \sigma_*(c_j^{-1} \cdot \tau_* c_k) = c_i^{-1} \cdot \sigma_* \tau_* c_k = c_i^{-1} \cdot (\sigma\tau)_* c_k$$

(ii) We prove $S_c(\tau) S_{\tau_* c}(\sigma) = S_c(\sigma\tau) = S_c(\sigma)\sigma_* S_c(\tau)$.
Set $\ell := \pi_{\sigma,\tau_* c}(k)$ and $i := \pi_{\tau,c}(\ell) = \pi_{\sigma\tau,c}(k)$. Then

$$c_i^{-1} \cdot \tau_* c_\ell \cdot (\tau_* c)_\ell^{-1} \cdot \sigma_*(\tau_* c)_k = c_i^{-1} \cdot \sigma_* \tau_* c_k = c_i^{-1} \cdot (\sigma\tau)_* c_k$$

□

Proposition

For any $c \in \widetilde{\mathcal{C}}$, the map $S_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ is injective.

Proof.

$\widetilde{\mathcal{B}}$ acts on $\widetilde{\mathcal{C}}$ freely and transitively. □

Digression: the Magnus cocycle

$\widetilde{\mathcal{B}}$ acts on $\Gamma = \pi_1(\mathbb{D} \setminus Z; z_0)$ free on m generators $g_{1,c}, \ldots, g_{m,c}$

Definition

Given $c \in \widetilde{\mathcal{C}}$, the Magnus cocycle $M_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ is

$$M_c(\sigma) := \left(\overline{\frac{\partial \sigma_* g_{j,c}}{\partial g_{i,c}}} \right)_{i,j=1,\ldots,m}$$

Fox calculus for a free group Γ with basis $g_1, \ldots, g_m$

- derivation $d : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ is additive homomorphism s.t.

  $$d(gh) = d(g)\cdot\varepsilon(h) + g\cdot d(h), \quad \varepsilon : \mathbb{Z}[\Gamma] \to \mathbb{Z} \text{ augmentation}$$

- $d(1) = 0$, $d(g^{-1}) = -g^{-1} d(g)$
- $\exists!$ derivation $\frac{\partial}{\partial g_i}$ such that $\frac{\partial g_j}{\partial g_i} = \delta_i^j$
- example: $\frac{\partial(g_{i-1} g_i g_{i-1}^{-1})}{\partial g_{i-1}} = 1 + g_{i-1} \frac{\partial(g_i g_{i-1}^{-1})}{\partial g_{i-1}} = 1 - g_{i-1} g_i g_{i-1}^{-1}$

The Magnus cocycle – continued

Proposition

Any two cocycles M_c and $M_{\tau_* c}$ are cohomologous:

$$M_{\tau_* c}(\sigma) = M_c(\tau)^{-1} M_c(\sigma) \sigma_* M_c(\tau), \quad \forall \sigma \in \widetilde{\mathcal{B}}.$$

Proof.

$$[M_c(\tau) M_{\tau_* c}(\sigma)]_{ik} = \sum_j \overline{\frac{\partial \tau_* g_{j,c}}{\partial g_{i,c}}} \cdot \overline{\frac{\partial \sigma_* \tau_* g_{k,c}}{\partial \tau_* g_{j,c}}} = \overline{\frac{\partial (\sigma\tau)_* g_{k,c}}{\partial g_{i,c}}} = M_c(\sigma\tau)_{ik}$$

□

Interesting (?) question: study $H^1(\widetilde{\mathcal{B}}, \mathrm{GL}_m(\mathbb{Z}[\Gamma]))$.

Magnus vs. Lefschetz monodromy cocycles - first comparison

$$M_c(\sigma_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 \\ 0 & \Box & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}, \quad S_c(\sigma_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 \\ 0 & \Box & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}$$

$$\Box = \begin{pmatrix} 1 - g_{i-1,c} g_{i,c}^{-1} g_{i-1,c}^{-1} & 1 \\ g_{i-1,c}^{-1} & 0 \end{pmatrix}, \qquad \Box = \begin{pmatrix} 0 & 1 \\ g_{i-1,c}^{-1} & 0 \end{pmatrix}$$

$$M_c(\varepsilon_{i,c}) = \mathbb{1}_m, \quad S_c(\varepsilon_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-1} & 0 & 0 \\ 0 & g_{i,c} & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}$$

View $\mathcal{B} \hookrightarrow \widetilde{\mathcal{B}}$ using framing determined by trivialization of $T\mathbb{D}$
$\rightsquigarrow$ restriction of the Magnus cocycle $M_c|_{\mathcal{B}}$ is injective

The Burau representation

Recall action of $\widetilde{\mathcal{B}}$ on Γ: $(\sigma_i)_* : \begin{cases} g_{i-1} \mapsto g_{i-1} g_i g_{i-1}^{-1}, \\ g_i \mapsto g_{i-1}, \\ g_j \mapsto g_j, \ j \neq i-1, i \end{cases}$

$(\varepsilon_i)_* = \mathrm{Id}$

Definition

For $c \in \widetilde{\mathcal{C}}$, the Burau representation $\overline{M}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}])$ is obtained by reducing the Magnus cocycle

$$\overline{M}_c(\sigma) := M_c(\sigma)\big|_{g_{1,c} = \cdots = g_{m,c} = t^{-1}}$$

Remarks on $\overline{M}_c$:

- independent of $c \in \widetilde{\mathcal{C}}$ up to conjugation
- reducible $1 + (m-1)$, with eigenvector $(1, t, t^2, \ldots, t^{m-1})$
- defines polynomial link invariant: Alexander polynomial (invariance under Markov moves)

The Monodromy representation

Definition (MOS)

For $c \in \widetilde{\mathcal{C}}$, the Lefschetz monodromy representation

$$\overline{S}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}])$$

is obtained by reducing the Lefschetz monodromy cocycle

$$\overline{S}_c(\sigma) := S_c(\sigma)\big|_{g_{1,c}=\cdots=g_{m,c}=t^{-1}}$$

Note: independent of $c \in \widetilde{\mathcal{C}}$ up to conjugation

does not define polynomial link invariant

Comparison of the Burau and Monodromy representations

$$\overline{M}_c(\sigma_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}, \quad \overline{S}_c(\sigma_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}$$

$$\square = \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}, \qquad \square = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

$$\overline{M}_c(\varepsilon_{i,c}) = \mathbb{1}_m, \quad \overline{S}_c(\varepsilon_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-1} & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ \mathrm{tr}(\mathrm{Id})}}{m} \neq \underset{\substack{\uparrow \\ \mathrm{tr}(\overline{S}_c(\sigma_i))}}{m-2} \neq \underset{\substack{\uparrow \\ \mathrm{tr}(\overline{M}_c(\sigma_i))}}{m-1-t} \neq m$$

Corollary

The Magnus class $[M_c]$ and the Lefschetz monodromy class $[S_c]$ are non-trivial and distinct.

Irreducibility of the monodromy representation

Proposition

The representation $\overline{S}_c$ is irreducible.

Proof.

Step 1. $\overline{S}_c$ cannot fix a subspace of dimension 1.

$$\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{cases} b = \lambda a \\ ta = \lambda b \end{cases} \Rightarrow \begin{cases} ta = \lambda^2 a \\ a, b = 0 \end{cases}$$

Step 2. $\overline{S}_c$ cannot be reducible $m = k + \ell$, $k, \ell \geq 2$. Assuming this to be true, we further specialize $t = 1$, and $\overline{S}_c$ descends to the canonical representation $\mathfrak{S}_m \to \mathrm{GL}_m(\mathbb{Z})$ which is reducible $m = 1 + (m-1)$. The $(m-1)$-dim. factor is the restriction to $x_1 + \cdots + x_m = 0$ and is irreducible, a contradiction. □

The monodromy representation and linking numbers

View $\mathcal{B} \hookrightarrow \widetilde{\mathcal{B}}$ via trivialization of $T\mathbb{D} \rightsquigarrow S_c : \mathcal{B} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$

Proposition

Let $\sigma \in \mathcal{PB} \subset \mathcal{B}$ be a pure braid. Then

$$S_c(\sigma) = \mathrm{Diag}(t^{\ell_1}, \dots, t^{\ell_m}), \quad \ell_i = \sum_{j>i} \mathrm{lk}(i,j) + \sum_{j<i} \mathrm{lk}(j,i),$$

$\mathrm{lk}(i,j) =$ *linking number of components i, j of the closed braid.*

Proof: direct computation using presentation of $\mathcal{PB}$ and

$$S_c(\sigma_{i,c}^2) = \mathrm{Diag}(1, \dots, \underset{i-1}{t}, \underset{i}{t}, \dots, 1)$$

□

Part II

Lefschetz fibrations and Morse theory

Lefschetz fibrations - local model

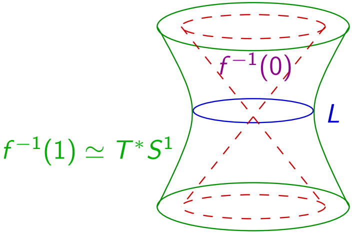
$$f : \mathbb{C}^{n+1} \to \mathbb{C}, \; f(z) = z_1^2 + \cdots + z_{n+1}^2$$

$$f^{-1}(1) = \{x + iy \; : \; |x|^2 - |y|^2 = 1, \; \langle x, y \rangle = 0\}$$

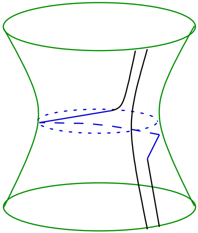
$$||| \qquad (x, y) \mapsto (x/|x|, |x|y) \quad \text{symplectic}$$

$$T^*S^n = \{(\xi, \eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \; : \; |\xi| = 1, \; \langle \xi, \eta \rangle = 0\}$$

Lefschetz fibrations - local model



$f(z_1, z_2) = z_1^2 + z_2^2$



Dehn twist along L

Monodromy around $t \mapsto e^{2\pi i t}$ = Dehn twist.

Vanishing cycle at 0 in the direction $\mathbb{R}^{\geq 0}$ is the 0-section S^n

self-intersection number is $2(-1)^{n/2}$ for even n, 0 for odd n

Lefschetz fibrations

Definition

A Lefschetz fibration over the disc $\mathbb{D} \subset \mathbb{C}$ is a holomorphic map $f : X \to \mathbb{D}$ with nondegenerate critical points, and which correspond to distinct critical values in $\mathring{\mathbb{D}}$.

X Kähler, $\dim X = n+1$, $m := \mathrm{card}(Z = \mathrm{Crit.val.}(f))$

Fix $z_0 \in \partial\mathbb{D}$ and $\{v_z \in T_z\mathbb{D}\}_{z \in Z}$. Choice of $c \in \widetilde{\mathcal{C}}$
$\rightsquigarrow$ vanishing cycles $L_{1,c}, \dots, L_{m,c} \subset M := f^{-1}(z_0)$
(local model + parallel transport by canonical connection)

Orientations for $L_{i,c} \rightsquigarrow$ monodromy character

$$\mathcal{N}_c^X : \Gamma = \pi_1(\mathbb{D} \setminus Z; z_0) \to \mathbb{Z}^{m \times m}$$

$$\mathcal{N}_c^X(g)_{ij} := \langle L_{i,c}, g_* L_{j,c} \rangle$$

The Lefschetz monodromy cocycle

Proposition (MOS)

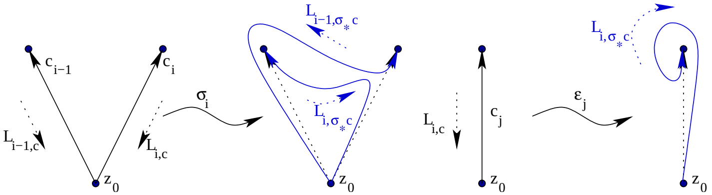
Given $\sigma \in \widetilde{\mathcal{B}}$ we have

$$\mathcal{N}^X_{\sigma_* c} = S_c(\sigma)^t \mathcal{N}^X_c S_c(\sigma).$$

(conjugate transpose + convolution product)

Proof.

We have $L_{i,\sigma_* c} = (c_{i'}^{-1} \cdot \sigma_* c_i)_*^{-1} L_{i',c}$, $i' = \pi_{\sigma,c}(i)$ etc. □



The Picard–Lefschetz formula

Recall $f : X \to \mathbb{D}$ Lefschetz fibration, $z_0 \in \partial\mathbb{D}$, $M := f^{-1}(z_0)$

$c \in \widetilde{\mathcal{C}} \rightsquigarrow$ loops $g_{1,c}, \dots, g_{m,c} \in \Gamma \rightsquigarrow (g_{i,c})_* \in \mathrm{Aut}(H_n(M))$

Proposition (Picard–Lefschetz formula)

Setting $\varepsilon = (-1)^{n(n+1)/2}$, *we have*

$$(g_{i,c})_*\alpha = \alpha - \varepsilon\langle L_{i,c}, \alpha\rangle L_{i,c}, \qquad \forall \alpha \in H_n(M).$$

- $\mathcal{N}_c^X(gg_{i,c}h) = \mathcal{N}_c^X(gh) - \varepsilon\mathcal{N}_c^X(g)E_i\mathcal{N}_c^X(h)$
- since Γ is free on the $g_{i,c}$'s, the map $\mathcal{N}_c^X : \Gamma \to \mathbb{Z}^{m\times m}$ is uniquely (and explicitly) determined by the matrix

$$N_c^X := \mathcal{N}_c^X(1)$$

Nonlinear monodromy cocycles

The matrix $N_c = (n_{ij}) = N_c^X$ has the following properties.

- $N_c^T = (-1)^n N_c$
- $n_{ii} = \begin{cases} 0, & n \text{ odd}, \quad \text{(anti-symmetric)} \\ 2 \cdot (-1)^{n/2}, & n \text{ even}. \quad \text{(symmetric)} \end{cases}$

Let $\mathcal{S}_m = \mathcal{S}_m(n) =$ the set of such matrices $\subset \mathrm{Mat}_m(\mathbb{Z})$.

Proposition (Bondal,MOS)

For each $c \in \widetilde{\mathcal{C}}$ there is a map $S_c : \widetilde{\mathcal{B}} \times \mathcal{S}_m \to \mathrm{GL}_m(\mathbb{Z})$ such that

$$N_{\sigma_* c} = S_c(\sigma, N_c)^T N_c S_c(\sigma, N_c), \quad \forall \sigma \in \widetilde{\mathcal{B}},$$

and moreover $S_c(\sigma\tau, N_c) = S_c(\tau, N_c) S_c(\sigma, N_{\tau_* c})$.

Conclusion: the orbit of N_c under conjugation with $S_c(\cdot, N_c)$ is an invariant of f.

Straight lines

Previous procedure: consider intersection numbers of vanishing cycles in the distinguished fiber $M = f^{-1}(z_0)$.

Alternative procedure: consider intersection number of two vanishing cycles L_i, L_j along some arbitrary path γ_{ij} joining the corresponding critical values $\rightsquigarrow$ collection of numbers $m_{ij} \in \mathbb{Z}$

Particular case: γ_{ij} = straight line segment $\rightsquigarrow$ numbers $\ell_{ij} \in \mathbb{Z}$.

Straight lines - continued

Proposition (MOS)

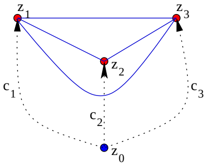
The collection (ℓ_{ij}) of intersection numbers along straight lines determines uniquely the matrices N_c, $c \in \widetilde{\mathcal{C}}$ via the Picard–Lefschetz formula.

Proof.

In the figure, we have

$$\ell_{13} = \langle L_1, (g_2)_* L_3 \rangle = \langle L_1, L_3 - \varepsilon \langle L_2, L_3 \rangle L_2 \rangle = n_{13} - \varepsilon n_{12} n_{23}.$$

$+$ induction argument □

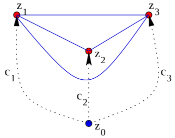


A dictionary

$\mathbb{R}$	$\mathbb{C}$
Morse function $f : M \to \mathbb{R}$	Lefschetz fibration $f : X \to \mathbb{C}$
$\{f \leq c - \rho\} \rightsquigarrow \{f \leq c + \rho\}$	monodromy of loop $c + \rho e^{2\pi it}$
unstable mfd. $W^u(x, -\nabla f)$	unstable mfd. $W^u(x, -\nabla \mathrm{Re}(e^{-i\theta} f))$
connecting trajectories $\mathcal{M}(x, y; -\nabla f)$	connecting trajectories $\mathcal{M}(x, y; -\nabla \mathrm{Re}(e^{-i\theta} f))$ $\theta = \mathrm{Arg}(f(y) - f(x))$
$\#\mathcal{M}(x, y; -\nabla f)$	$L_x \cdot L_y$ along segment $[f(x), f(y)]$
Morse differential $\partial x = \sum_y \#\mathcal{M}(x, y)y$	monodromy character $\mathcal{N} \equiv \mathcal{N}(1) \equiv$ collection $\{\ell_{xy}\}$
$\partial \circ \partial = 0$	Picard-Lefschetz formula $\ell_{13} = n_{13} - \varepsilon n_{12} n_{23}$

Homology

$\mathbb{R}$	$\mathbb{C}$
Morse homology $\simeq H_*(X)$	$\widetilde{\mathcal{B}}$-orbit of N
invariant under deformations f_t through smooth functions	invariant under deformations f_t through Lefschetz fibrations



Example of an invariant of the $\widetilde{\mathcal{B}}$-orbit of N:

$(\mathbb{Z}^m / \ker N, N)$ free abelian group + nondeg. bilinear form

Our hope of a dictionary

$\mathbb{R}$ exact mfd. $(M, \omega = d\lambda)$	$\mathbb{C}$ exact mfd. $(X, I, \omega^\theta = d\lambda)$
Floer theory = Morse th. action functional on loop space	Floer theory = Picard–Lefschetz th. holomorphic action functional

The End

Thank you for your attention and patience!