The Cyclotomic Birman-Murakami-Wenzl Algebras and Cylindrical Tangles

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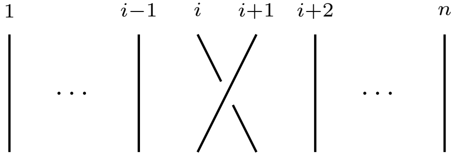
Braid Groups

The Braid group on n strings:

$$\mathcal{B}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall\ i = 1, \ldots, n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2 \end{array} \right\rangle$$

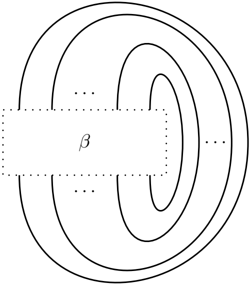
$\sigma_i \quad =$



The braid group $\mathcal{B}_n$ is an Artin group of type A.

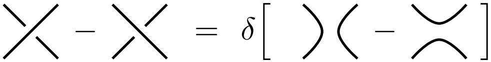
Closure of Braids

Taking closure of braids leads to knots and links in S^3



→ link invariants

→ Kauffman link polynomial



→ **Birman-Murakami-Wenzl (BMW) Algebras**

Definition

R - commutative ring with 1.
units $A_0, q, \lambda \in R$ such that $\lambda - \lambda^{-1} = \delta(1 - A_0)$,
where $\delta := q - q^{-1}$.
The **BMW algebra** $\mathscr{C}_n$ is defined by
Generators: $X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}$ and $e_1, \ldots, e_{n-1}$

Relations:

$$\begin{array}{rcll}
X_i - X_i^{-1} & = & \delta(1 - e_i) & \\
X_i X_j & = & X_j X_i & \text{for } |i-j| \geq 2 \\
X_i X_{i+1} X_i & = & X_{i+1} X_i X_{i+1} & \\
X_i e_j & = & e_j X_i & \text{for } |i-j| \geq 2 \\
e_i e_j & = & e_j e_i & \text{for } |i-j| \geq 2 \\
X_i e_i & = & e_i X_i \;=\; \lambda e_i & \\
X_i X_j e_i & = & e_j e_i \;=\; e_j X_i X_j & \text{for } |i-j| = 1 \\
e_i e_{i\pm 1} e_i & = & e_i & \\
e_i^2 & = & A_0 e_i &
\end{array}$$

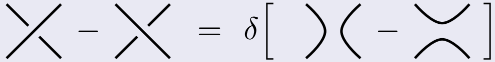
Kauffman Tangle Algebras

Definition

$\mathbb{T}_n :=$ set of n-tangles up to regular isotopy.

$\mathbb{KT}_n :=$ monoid R-algebra $R[\mathbb{T}_n]$ modulo the following relations:

(1) (Kauffman skein relation)



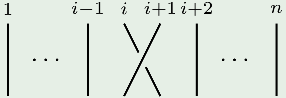
(2) (Untwisting relation)

(3) $T \amalg \bigcirc = A_0\, T$,
where $T \amalg \bigcirc$ is the union of T and a circle which has no crossings with T or itself.

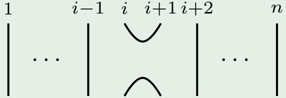
Theorem (Morton and Wassermann)

1. The BMW algebra $\mathscr{C}_n$ is isomorphic to $\mathbb{KT}_n$.

$X_i \longmapsto$



$e_i \longmapsto$



2. The BMW algebra $\mathscr{C}_n$ is R-free of rank $(2n-1)!! = (2n-1)\cdot(2n-3)\cdots 3\cdot 1 = \frac{(2n)!}{2^n n!}$.
3. It has a nice diagrammatic basis that is easy to write down.

Diagrammatic basis of $\mathscr{C}_n$

- BMW algebra is a deformation ("q-analogue") of the Brauer algebra, in the same way the Iwahori-Hecke algebra of type A is a deformation of the group algebra of the symmetric group.

  [Morton & Wassermann, Halverson & Ram]

$$\text{Brauer} \xrightarrow{\text{impose over and under crossings}} \text{BMW}$$

- Alternatively, the Brauer algebra is the "classical limit" (send $q \mapsto 1$) of the BMW algebra or Kauffman Tangle algebra;

$$\text{BMW} \xrightarrow{\text{over} = \text{under crossings}} \text{Brauer}$$

  (So diagrams with only vertical strands degenerate into permutations.)

FACTS continued

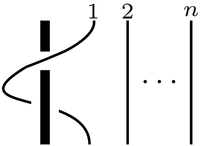
1. Iwahori-Hecke algebra associated with the symmetric group $\mathfrak{S}_n$ is a quotient of $\mathscr{C}_n$.
2. It is cellular, in the sense of Graham and Lehrer, [Enyang, Xi]
3. and quasi-hereditary, in the sense of Cline, Parshall and Scott. [Xi]
4. The Braid group embeds in the BMW algebra: $\mathcal{B}_n \hookrightarrow \mathscr{C}_n$. [Bigelow], [Krammer], [Zinno]

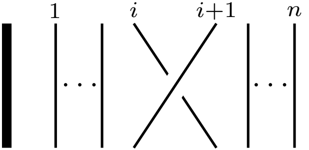
Artin group of type B

The Artin group of type B_n:

$$\left\langle s_0, s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{cl} s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 & \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \forall\, 1 \le i \le n-2 \\ s_i s_j = s_j s_i & \text{if } |i-j| \ge 2 \end{array} \right\rangle$$

Type B braids: affine or cylindrical braids.

$s_0 =$ 

$s_i =$ 

These are ordinary braids on $n+1$ strings where the first string is pointwise fixed. The leftmost string is drawn as a vertical line segment and other strings may twist about it.
Taking the closure of these braids leads to the study of links in the solid torus.

Definition

R - commutative ring with 1. Units A_0, q_0, q, λ and further elements $q_1 \ldots, q_{k-1}$, $A_1, \ldots, A_{k-1}$ s.t. $\lambda - \lambda^{-1} = \delta(1 - A_0)$. The **cyclotomic BMW algebra** $\mathscr{B}_n^k$ is defined by

Generators: $Y^{\pm 1}, X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}$ and $e_1, \ldots, e_{n-1}$

Relations:

$$
\begin{array}{rcll}
X_i - X_i^{-1} & = & \delta(1 - e_i) & \\
X_i X_j & = & X_j X_i & \text{for } |i-j| \geq 2 \\
X_i X_{i+1} X_i & = & X_{i+1} X_i X_{i+1} & \\
X_i e_j & = & e_j X_i & \text{for } |i-j| \geq 2 \\
e_i e_j & = & e_j e_i & \text{for } |i-j| \geq 2 \\
X_i e_i & = & e_i X_i \; = \; \lambda e_i & \\
X_i X_j e_i & = & e_j e_i \; = \; e_j X_i X_j & \text{for } |i-j| = 1 \\
e_i e_{i\pm 1} e_i & = & e_i & \\
e_i^2 & = & A_0 e_i &
\end{array}
$$

Definition (continued)

$$
\begin{aligned}
Y^k &= \sum_{i=0}^{k-1} q_i Y^i && \\
X_1 Y X_1 Y &= Y X_1 Y X_1 && \\
Y X_i &= X_i Y && \text{for } i > 1 \\
Y e_i &= e_i Y && \text{for } i > 1 \\
Y X_1 Y e_1 &= \lambda^{-1} e_1 = e_1 Y X_1 Y && \\
e_1 Y^m e_1 &= A_m e_1 && \text{for } 0 \le m \le k-1
\end{aligned}
$$

Questions

Questions

1. Is it always free as an R-module? What is the rank?
2. Is $\mathscr{B}_n^k$ isomorphic to some affine/cylindrical analogue of the Kauffman Tangle Algebra?

Answer is not always!

The k^{th} order relation on Y creates torsion on e_1. This implies we need to impose further restrictions on the parameters of our ground ring

$\rightsquigarrow$ rings with "admissible" parameters.

To determine the precise form of these conditions, one needs to focus on the representation theory of $\mathscr{B}_2^k$. [Wilcox & Y]

Admissibility conditions

Definition

The family of parameters $(A_0, \ldots, A_{k-1}, q_0, \ldots, q_{k-1}, q, \lambda)$ is called **admissible** if

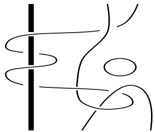
$$\beta = h_0 = h_1 = \ldots = h_{z-\epsilon} = h'_1 = h'_2 = \ldots = h'_{z-\epsilon} = 0.$$

$$\begin{aligned}
\beta &= q_0\lambda - q_0^{-1}\lambda^{-1} + (1-\epsilon)\delta \\
h_0 &= \lambda - \lambda^{-1} + \delta(A_0 - 1) \\
h_l &= \lambda^{-1}(q_l + q_0^{-1}q_{k-l}) + \delta\left[\sum_{r=1}^{k-l} q_{r+l}A_r - \sum_{i=\max(l+1,z)}^{\lfloor \frac{l+k}{2} \rfloor} q_{2i-l} + \sum_{i=\lceil \frac{l}{2} \rceil}^{\min(l,z-1)} q_{2i-l}\right] \\
h'_l &= \sum_{r=1}^{l} q_0^{-1} q_{r+k-l} A_r - \sum_{r=0}^{k-l} q_{r+l} A_r \\
&\quad - \sum_{i=\lceil \frac{l}{2} \rceil}^{l-1} (q_0^{-1} q_{k-2i+l} + q_{2i-l}) + \sum_{i=z}^{\lfloor \frac{l+k}{2} \rfloor} (q_0^{-1} q_{k-2i+l} + q_{2i-l}).
\end{aligned}$$

- Originally, "admissibility" is a set of conditions on the parameters in R which ensure that $\mathscr{B}_2^k$ is of rank $3k^2$.
- In fact, these conditions turns out to be sufficient for freeness results for any n.
- It is not straightforward to show that there are any non-trivial rings with admissible parameters; i.e. that the conditions are consistent with each other.
- Admissibility is very close to essentially a set of conditions required for a (nondegenerate) Markov trace function on $\mathscr{B}_n^k$ (required later).

Cyclotomic Kauffman Tangle Algebras

Affine 2-tangle diagram:



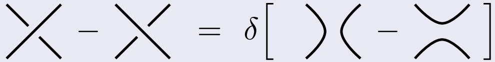
Definition

$\widehat{\mathbb{T}}_n$:= monoid of regular isotopy equivalence classes of affine n-tangles.

$\mathbb{KT}_n^k$:= monoid S-algebra $S[\widehat{\mathbb{T}}_n]$ modulo the following relations:

(1) (cyclotomic skein relation) $$\sum_{r=0}^{k} q_r \, \mathcal{Y}^r = 0.$$

(2) (Kauffman skein relation)



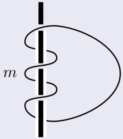
Definition (continued)

(3) (Untwisting relation)

 $= \lambda$ and  $= \lambda^{-1}$

(4) (Free loop relations)

For all $m = 0, \ldots, k-1$,

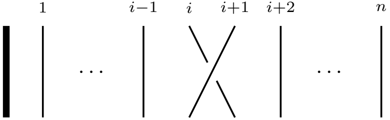
$T \amalg$  $= A_m\, T$

Theorem

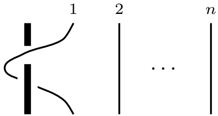
Let S be a ring with admissible parameters.

1. The cyclotomic BMW algebra $\mathscr{B}_n^k$ is isomorphic to the cyclotomic Kauffman Tangle algebra $\mathbb{KT}_n^k$.
2. $\mathscr{B}_n^k$ is S-free of rank $k^n(2n-1)!!$.

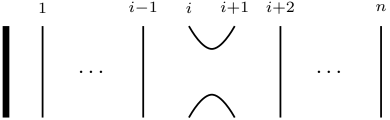
$X_i \longmapsto$



$Y \longmapsto$



$e_i \longmapsto$



Ingredients for proof

- Spanning - lots of tedious calculations.
- Linear independence:
  - admissibility.
  - Nondegenerate Markov trace on $\mathscr{B}_n^k \sim$ closure of affine tangles. (Alternatively, normal form for $\mathscr{B}_n^k$ gives an inductive definition of trace)

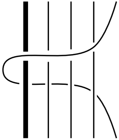
Goodman & Hauschild-Mosley and Rui, Xu & Si have also established freeness of $\mathscr{B}_n^k$.

G&H work over a more restrictive class of rings (stronger notion of admissibility plus ID). Their proof involves topological arguments (spanning) and Jones Basic Construction theory (semisimple setting).

R&X&S work with a different stronger notion of admissibility too. Their proof involves constructing seminormal representations.

Basis for $\mathscr{B}_n^k$

$$Y_i' := X_{i-1}X_{i-2}\ldots X_1 Y X_1 \ldots X_{i-2}X_{i-1}.$$

$Y_4' =$ 

Suppose $l \geq 1$, $i \leq j \leq l$ and $p \in \mathbb{Z}$.
$\alpha_{ijl}^p := Y_i'^{\,p} X_i \ldots X_{j-1} e_j \ldots e_l.$

The affine 7-tangle associated with the element $\alpha_{4,6,6}^0 \alpha_{1,3,4}^0 \in \mathscr{B}_7^k$.

Basis for $\mathscr{B}_n^k$

$^* : \mathscr{B}_n^k \to \mathscr{B}_n^k$ is the anti-involution which fixes all generators.

$\widetilde{\mathfrak{W}}_{n-2f,\,k} =$ certain 'lifting' of a basis $X_{n-2f,k}$ of the smaller Ariki-Koike algebra $\mathfrak{h}_{n-2f,k}$ into $\mathscr{B}_n^k$.

Theorem

The set of all elements of this form is a basis of $\mathscr{B}_n^k$ over S.

$$\alpha^{s_1}_{i_1 j_1, n-1} \cdots \alpha^{s_f}_{i_f j_f, n-2f+1} \chi^{(n-2f)} (\alpha^{t_f}_{g_f h_f, n-2f+1})^* \cdots (\alpha^{t_1}_{g_1 h_1, n-1})^*,$$

where $f = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, $i_1 > i_2 > \ldots > i_f$, $g_1 > g_2 > \ldots > g_f$ and, for each $m = 1, 2, \ldots f$, we require $1 \leq i_m < j_m \leq n - 2m + 1$, $s_1, \ldots, s_f, t_1, \ldots, t_f \in \left\{ \left\lfloor \frac{k}{2} \right\rfloor - (k-1), \ldots, \left\lfloor \frac{k}{2} \right\rfloor \right\}$ and $\chi^{(n-2f)}$ is an element of $\widetilde{\mathfrak{W}}_{n-2f,\,k}$.

(Remark: This reduces to a basis of $\mathscr{C}_n$ when $k = 1$)

Examples of cellular algebras:

- Iwahori-Hecke algebras of the symmetric group. [Graham & Lehrer]
- Brauer and Temeperley-Lieb algebras. [Graham & Lehrer]
- BMW algebras. [Enyang, Xi]
- cyclotomic Brauer algebras. [Rui]
- Ariki-Koike algebras [Graham & Lehrer] & [Dipper, James & Mathas]

Question: Is $\mathscr{B}_n^k$ cellular?

Theorem

Let S be a ring with split admissible parameters. Then $\mathscr{B}_n^k(S)$ is a **cellular** algebra, in the sense of Graham and Lehrer.

Analysing the representation theory of $\mathscr{B}_n^k$:
Cellularity

- $\rightsquigarrow$ parametrisation of irreducible $\mathscr{B}_n^k$-modules.
- $\rightsquigarrow$ We know how to construct all irreps in generic case.
- $\rightsquigarrow$ semisimplicity criterion. [Rui & Si]

Current/Future Directions

- Knot theory:
  Nondegenerate Markov trace on $\mathscr{B}_n^k$
  $\rightsquigarrow$ Kauffman-type invariants of links in the solid torus.

- Other types of Artin groups? [Cohen, Gijsbers & Wales]
  Other types of complex reflection groups?
  Associated embedding and linearity questions?

  [Bigelow], [Krammer], [Zinno], [Digne], [Cohen & Wales], [Castella], [Marin], ...

- Statistical Mechanics. $(A_0 = 0)$

- Quantum groups.

- Schur-Weyl Duality. [Orellana & Ram]