The Cyclotomic Birman-Murakami-Wenzl
Algebras
and
Cylindrical Tangles

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The Braid group on $n$ strings:

$$\mathcal{B}_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \forall \ i = 1, \ldots, n-2 \rangle$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2$$

The braid group $\mathcal{B}_n$ is an Artin group of type $A$. 
Taking closure of braids leads to knots and links in $S^3$

$\beta \cdots \rightarrow$ link invariants

$\rightarrow$ Kauffman link polynomial

$\times - \times = \delta \left[ \begin{array}{c}
\end{array} \right]$

$\rightarrow$ Birman-Murakami-Wenzl (BMW) Algebras
BMW Algebras

**Definition**

$R$ - commutative ring with 1.

units $A_0, q, \lambda \in R$ such that $\lambda - \lambda^{-1} = \delta(1 - A_0)$,

where $\delta := q - q^{-1}$.

The **BMW algebra** $C_n$ is defined by

**Generators:** $X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}$ and $e_1, \ldots, e_{n-1}$

**Relations:**

\begin{align*}
X_i - X_i^{-1} &= \delta(1 - e_i) \\
X_i X_j &= X_j X_i & \text{for } |i - j| \geq 2 \\
X_i X_{i+1} X_i &= X_{i+1} X_i X_i & \\
X_i e_j &= e_j X_i & \text{for } |i - j| \geq 2 \\
e_i e_j &= e_j e_i & \text{for } |i - j| \geq 2 \\
X_i e_i &= e_i X_i = \lambda e_i \\
X_i X_j e_i &= e_j e_i = e_j X_i X_j & \text{for } |i - j| = 1 \\
e_i e_{i \pm 1} e_i &= e_i \\
e_i^2 &= A_0 e_i
\end{align*}
\( \mathbb{T}_n := \text{set of } n\text{-tangles up to regular isotopy.} \)

\( \mathbb{KT}_n := \text{monoid } R\text{-algebra } R[\mathbb{T}_n] \text{ modulo the following relations:} \)

1. **(Kauffman skein relation)**
   \[
   \begin{array}{c}
   \includegraphics{kauffman_skein_relation.png} \\
   \end{array}
   \]
   \[
   \delta \left[ \begin{array}{c}
   \includegraphics{kauffman_skein_relation.png} \\
   \end{array} \right] = \begin{array}{c}
   \includegraphics{kauffman_skein_relation.png} \\
   \end{array}
   \]

2. **(Untwisting relation)**
   \[
   \begin{array}{c}
   \includegraphics{untwisting_relation.png} \\
   \end{array}
   \]
   \[
   \begin{array}{c}
   \includegraphics{untwisting_relation.png} \\
   \end{array} = \lambda \\
   \begin{array}{c}
   \includegraphics{untwisting_relation.png} \\
   \end{array} = \lambda^{-1}
   \]

3. **(T ⊔ ∅ = A_0 T)**
   where \( T \cup \bigcirc \) is the union of \( T \) and a circle which has no crossings with \( T \) or itself.
Theorem (Morton and Wassermann)

1. The BMW algebra $C_n$ is isomorphic to $KT_n$.

$$X_i \mapsto \ \begin{array}{cccccc}
1 & i-1 & i & i+1 & i+2 & n \\
\ldots & \times & \ldots & & & \\
\end{array}$$

$$e_i \mapsto \ \begin{array}{cccccc}
1 & i-1 & i & i+1 & i+2 & n \\
\ldots & \cup & \ldots & & & \\
\end{array}$$

2. The BMW algebra $C_n$ is $R$-free of rank 
   $(2n - 1)!! = (2n - 1) \cdot (2n - 3) \cdots 3 \cdot 1 = \frac{(2n)!}{2^n n!}.$

3. It has a nice diagrammatic basis that is easy to write down.
BMW algebra is a \textit{deformation} ("q-analogue") of the \textit{Brauer algebra}, in the same way the Iwahori-Hecke algebra of type $A$ is a deformation of the group algebra of the symmetric group. 

\cite{Morton & Wassermann, Halverson & Ram}

\begin{align*}
\text{Brauer} & \quad \text{impose over and under crossings} \quad \rightarrow \quad \text{BMW} \\
\text{BMW} & \quad \text{over} = \text{under crossings} \quad \rightarrow \quad \text{Brauer}
\end{align*}

Alternatively, the Brauer algebra is the "classical limit" (send $q \mapsto 1$) of the BMW algebra or Kauffman Tangle algebra; 

(\text{So diagrams with only vertical strands degenerate into permutations.})
1. Iwahori-Hecke algebra associated with the symmetric group $S_n$ is a quotient of $\mathcal{C}_n$.

2. It is cellular, in the sense of Graham and Lehrer, [Enyang, Xi]

3. and quasi-hereditary, in the sense of Cline, Parshall and Scott. [Xi]

4. The Braid group embeds in the BMW algebra: $\mathcal{B}_n \hookrightarrow \mathcal{C}_n$. [Bigelow], [Krammer], [Zinno]
The Artin group of type $B_n$:

\[
\langle s_0, s_1, \ldots, s_{n-1} \mid s_0s_1s_0s_1 = s_1s_0s_1s_0, \\
    s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \\
    s_is_j = s_js_i \quad \forall 1 \leq i \leq n-2, \\
    \text{if } |i-j| \geq 2 \rangle
\]

Type $B$ braids: affine or cylindrical braids.

These are ordinary braids on $n+1$ strings where the first string is pointwise fixed. The leftmost string is drawn as a vertical line segment and other strings may twist about it. Taking the closure of these braids leads to the study of links in the solid torus.
Cyclotomic BMW Algebras

Definition

$R$ - commutative ring with 1. Units $A_0, q_0, q, \lambda$ and further elements $q_1 \ldots, q_{k-1}, A_1, \ldots, A_{k-1}$ s.t. $\lambda - \lambda^{-1} = \delta(1 - A_0)$.

The **cyclotomic BMW algebra** $B_n^k$ is defined by

Generators: $Y^{\pm 1}, X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}$ and $e_1, \ldots, e_{n-1}$

Relations:

\[
X_i - X_i^{-1} = \delta(1 - e_i)
\]

\[
X_iX_j = X_jX_i \quad \text{for } |i - j| \geq 2
\]

\[
X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1}
\]

\[
X_i e_j = e_j X_i \quad \text{for } |i - j| \geq 2
\]

\[
e_i e_j = e_j e_i \quad \text{for } |i - j| \geq 2
\]

\[
X_i e_i = e_i X_i = \lambda e_i
\]

\[
X_i X_j e_i = e_j e_i = e_j X_i X_j \quad \text{for } |i - j| = 1
\]

\[
e_i e_{i \pm 1} e_i = e_i
\]

\[
e_i^2 = A_0 e_i
\]
Definition (continued)

\[
\begin{align*}
Y^k &= \sum_{i=0}^{k-1} q_i Y^i \\
X_1 YX_1 Y &= YX_1 YX_1 \\
YX_i &= X_i Y \quad \text{for } i > 1 \\
Ye_i &= e_i Y \quad \text{for } i > 1 \\
YX_1 Ye_1 &= \lambda^{-1} e_1 = e_1 YX_1 Y \\
e_1 Y^m e_1 &= A_m e_1 \quad \text{for } 0 \leq m \leq k - 1
\end{align*}
\]
Questions

1. Is it always free as an $R$-module? What is the rank?

2. Is $B^k_n$ isomorphic to some affine/cylindrical analogue of the Kauffman Tangle Algebra?

Answer is not always!

The $k^{th}$ order relation on $Y$ creates torsion on $e_1$. This implies we need to impose further restrictions on the parameters of our ground ring

$$\rightsquigarrow$$ rings with “admissible” parameters.

To determine the precise form of these conditions, one needs to focus on the representation theory of $B^k_2$. [Wilcox & Y]
Admissibility conditions

**Definition**

The family of parameters \((A_0, \ldots, A_{k-1}, q_0, \ldots, q_{k-1}, q, \lambda)\) is called **admissible** if

\[
\beta = h_0 = h_1 = \ldots = h_{z-\epsilon} = h'_1 = h'_2 = \ldots = h'_{z-\epsilon} = 0.
\]

\[
\beta = q_0\lambda - q_0^{-1}\lambda^{-1} + (1 - \epsilon)\delta
\]

\[
h_0 = \lambda - \lambda^{-1} + \delta(A_0 - 1)
\]

\[
h_l = \lambda^{-1}(q_l + q_0^{-1}q_{k-l}) + \delta \left[ \sum_{r=1}^{k-l} q_{r+l}A_r - \sum_{i=\max(l+1,z)}^{\lfloor \frac{l+k}{2} \rfloor} q_{2i-l} + \sum_{i=\lceil \frac{l}{2} \rceil}^{\min(l,z-1)} q_{2i-l} \right]
\]

\[
h'_l = \sum_{r=1}^{l} q_0^{-1}q_{r+k-l}A_r - \sum_{r=0}^{k-l} q_{r+l}A_r
\]

\[- \sum_{i=\lceil \frac{l}{2} \rceil}^{l-1} (q_0^{-1}q_{k-2i+l} + q_{2i-l}) + \sum_{i=z}^{\lfloor \frac{l+k}{2} \rfloor} (q_0^{-1}q_{k-2i+l} + q_{2i-l}).\]
Originally, “admissibility” is a set of conditions on the parameters in $R$ which ensure that $\mathcal{B}_2^k$ is of rank $3k^2$.

In fact, these conditions turn out to be sufficient for freeness results for any $n$.

It is not straightforward to show that there are any non-trivial rings with admissible parameters; i.e. that the conditions are consistent with each other.

Admissibility is very close to essentially a set of conditions required for a (nondegenerate) Markov trace function on $\mathcal{B}_n^k$ (required later).
Affine 2-tangle diagram:

Definition

\( \hat{T}_n := \text{monoid of regular isotopy equivalence classes of affine } n\text{-tangles.} \)

\( \text{KT}_n^k := \text{monoid } S\text{-algebra } S[\hat{T}_n] \text{ modulo the following relations:} \)

(1) (cyclotomic skein relation)

\[
\sum_{r=0}^{k} q^r Y^r = 0.
\]

(2) (Kauffman skein relation)

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}
= \delta \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) .. controls (0.5,0.5) and (0.5,-0.5) .. (1,0);
\end{tikzpicture}
\end{array}

- \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) .. controls (0.5,0.5) and (0.5,-0.5) .. (1,0);
\end{tikzpicture}
\end{array}
\]
Definition (continued)

(3) (Untwisting relation)

\[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{untwisting_relation.png}
\end{array} \]

\[ = \lambda \quad \text{and} \quad = \lambda^{-1} \]

(4) (Free loop relations)

For all \( m = 0, \ldots, k - 1, \)

\[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{free_loop_relation.png}
\end{array} \]

\[ T \quad \includegraphics[width=0.2\textwidth]{loop_relation.png} = A_m T \]
Theorem

Let $S$ be a ring with admissible parameters.

1. The cyclotomic BMW algebra $\mathcal{B}_n^k$ is isomorphic to the cyclotomic Kauffman Tangle algebra $\mathcal{KT}_n^k$.

2. $\mathcal{B}_n^k$ is $S$-free of rank $k^n(2n - 1)!!$. 

$\mathcal{B}_n^k \cong \mathcal{KT}_n^k$
Ingredients for proof

- Spanning - lots of tedious calculations.
- Linear independence:
  - admissibility.
  - Nondegenerate Markov trace on $B_n^k \sim$ closure of affine tangles. (Alternatively, normal form for $B_n^k$ gives an inductive definition of trace)

Goodman & Hauschild-Mosley and Rui, Xu & Si have also established freeness of $B_n^k$.

**G&H** work over a more restrictive class of rings (stronger notion of admissibility plus ID). Their proof involves topological arguments (spanning) and Jones Basic Construction theory (semisimple setting).

**R&X&S** work with a different stronger notion of admissibility too. Their proof involves constructing seminormal representations.
Basis for $B^k_n$

$$Y'_i := X_{i-1} X_{i-2} \ldots X_1 Y X_1 \ldots X_{i-2} X_{i-1}.$$  \hspace{2cm} Y'_4 = \begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}

Suppose $l \geq 1$, $i \leq j \leq l$ and $p \in \mathbb{Z}$.

$$\alpha^p_{ijl} := Y'^p_i X_i \ldots X_{j-1} e_j \ldots e_l.$$  \hspace{2cm} \begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}

The affine 7-tangle associated with the element $\alpha^0_{4,6,6} \alpha^0_{1,3,4} \in B^7_k$.  \hspace{2cm} \begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
Basis for $\mathcal{B}_n^k$

$*: \mathcal{B}_n^k \rightarrow \mathcal{B}_n^k$ is the anti-involution which fixes all generators.

$\widetilde{\mathcal{W}}_{n-2f,k}$ = certain ‘lifting’ of a basis $X_{n-2f,k}$ of the smaller Ariki-Koike algebra $\mathfrak{h}_{n-2f,k}$ into $\mathcal{B}_n^k$.

**Theorem**

The set of all elements of this form is a basis of $\mathcal{B}_n^k$ over $S$.

$$\alpha_{i_1j_1,n-1}^{s_1} \cdots \alpha_{i_fj_f,n-2f+1}^{s_f} \chi^{(n-2f)}(\alpha_{g_fh_f,n-2f+1}^{t_f})^* \cdots (\alpha_{g_1h_1,n-1}^{t_1})^*,$$

where $f = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, $i_1 > i_2 > \ldots > i_f$, $g_1 > g_2 > \ldots > g_f$ and, for each $m = 1, 2, \ldots f$, we require $1 \leq i_m < j_m \leq n - 2m + 1$, $s_1, \ldots, s_f, t_1, \ldots, t_f \in \{ \lfloor \frac{k}{2} \rfloor - (k - 1), \ldots, \lfloor \frac{k}{2} \rfloor \}$ and $\chi^{(n-2f)}$ is an element of $\widetilde{\mathcal{W}}_{n-2f,k}$.

*(Remark: This reduces to a basis of $\mathcal{C}_n$ when $k = 1$)*
Examples of cellular algebras:

- Iwahori-Hecke algebras of the symmetric group. [Graham & Lehrer]
- Brauer and Temeperley-Lieb algebras. [Graham & Lehrer]
- BMW algebras. [Enyang, Xi]
- cyclotomic Brauer algebras. [Rui]
- Ariki-Koike algebras [Graham & Lehrer] & [Dipper, James & Mathas]

**Question:** Is $B_n^k$ cellular?
Theorem

Let $S$ be a ring with split admissible parameters. Then $\mathcal{B}_n^k(S)$ is a cellular algebra, in the sense of Graham and Lehrer.

Analysing the representation theory of $\mathcal{B}_n^k$:
Cellularity

$\leadsto$ parametrisation of irreducible $\mathcal{B}_n^k$-modules.

$\leadsto$ We know how to construct all irreps in generic case.

$\leadsto$ semisimplicity criterion. [Rui & Si]
Current/Future Directions

- **Knot theory:**
  - Nondegenerate Markov trace on $B^k_n$
  - $\sim$ Kauffman-type invariants of links in the solid torus.

- **Other types of Artin groups?** [Cohen, Gijsbers & Wales]
  - Other types of complex reflection groups?
  - Associated embedding and linearity questions?
    - [Bigelow], [Krammer], [Zinno], [Digne], [Cohen & Wales], [Castella], [Marin], . . .

- **Statistical Mechanics.** ($A_0 = 0$)

- **Quantum groups.**

- **Schur-Weyl Duality.** [Orellana & Ram]